PHD

Rearrangements and Vortices

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Award date:
2014

Awarding institution:
University of Bath

Link to publication

Rearrangements and Vortices

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

26th March 2014

Acknowledgements

I wrote this thesis with the financial support and facilities provided by the University of Bath. I would like to thank Professor Geoffrey Burton for this excellent supervision, which was always expertly insightful. I really enjoyed Professor Burton's supervision.

I thank the examiners for taking the time and effort to read my thesis, and allowing me to defend it.

I am grateful to Professor Jay Sivaloganathan, Professor Valery Smyshylaev and all the other lecturers with whom I discussed my work and my career. I would like to thank the staff of the Department of Mathematics office, Maren Eckhoff, Matt Pressland and my other fellow mathematicians in my office 4W 2.19. It was really great to teach so many undergraduate students, and I wish you all well with your future studies and careers.

Samantha Fiveash, you are the most wonderful person in the world, and I could not have completed this thesis without your kindness, your support, your love. I thank you with all my heart.

Fraeya Whiffin, Stephen Bush, Berin Midmer, Phil Irvine and Adam Jackson, you have been great to debate with, particularly when discussing my research concerns over drinks in Parade. Our little club is still going strong, and I hope it does even when we all begin to depart from this fair university.

Cerian Jenkins, I have never derived more fun from writing than under your stewardship. I really hope that you obtain all that you ask from life.

Nick Hill, Gemma Isherwood, Simon Rushton, Ben Hooper, Tomos Evans, Helen Edworthy, Elliott Campbell, Poppy Peake, Holly Narey, Scarlett Clark, Siân Maria Morgan, Kat Agg and everyone else in Media, it was really excellent to work with you all, and I'm sure greatness – or at least, Happy Hour – awaits. In the name of synergy, I would like to thank Jonathan Archer, Tamar Hudson, Darius Gilani and Hollie Christian-Brookes for helping me get more involved with the other parts of Media.

Beth McKendrick, Jennifer Dixon, Aaron Forward, Jack Tuxford, Amy Jackson, Molly McParland, Joe Turnbull, Rosamund Herapath, Henry Rackley, Lily Morris, Julia

McKay, Scott Kemp, Timo Haber, Corinna Hornwall, Chris Gynn, Adam Morris, Alex Clarke, Natalie Donat and everyone else from the University of Bath Debating Society, I really enjoyed formal debating, and I will certainly be applying all my sharp debating skills on you and others in the future. It's a brilliant society, and I hope it continues to be so after I leave.

Richard Guthrie, Simon O'Kane, Kate McAtamney and Jason Leake, Thursday afternoons are always spent well over cups of coffee that I don't drink, and discussing topics with such expertise. I'm sure I may take breaks late on Thursdays in memory of the PGA Cuppa Coffee.

Ellie Hynes, Sally Williamson, Vladimira Dermendjieva, Scott Burfiend, Rhiannon Norfolk, Tom Janicot, Peter Hachfeld, Penelope Bielkcus, Katie Barnby, Jordan Kenny, Paul Goodstadt and everyone else competing in the University of Bath Students' Union election or sitting on the Elections Committee, those two weeks were a fantastic whirlwind, and I really hope we've all recovered by now. I do sometimes wonder if I have.

Hanna Wade, it was amazing to have such a compatriot on local housing issues, and you will undoubtedly go on to achieve great things in the future.

Charlie Slack, Ian Robinson, Tim Stoneman, Amy Young and everyone else in the Students' Union, thank you for enduring all my inquiries, and I'm sure representational work will continue to be just challenging and fun.

I apologise if I've missed anyone out by name, but my life has been enhanced by so many good people.

To the seagulls of Bath, I'll never get that sleep back.

Thanks to my parents and friends, without whom this thesis would have exhausted my mind, my patience and my abilities. There were times where I believed I was falling from the zeniths of my undergraduate degree, and you were always there to help sustain my course.

Contents

Chapter 1

Introduction

1.1 Abstract

Rearrangements are two measurable real-valued functions that have equal measure of pre-images of upper level sets. In this thesis, I will investigate several matters and problems relating to rearrangements:

- the relationship between assumptions on the measure space and desirable properties of the set of rearrangements, and the validity of rearrangement inequalities;
- generalising the Mountain Pass Lemma over rearrangements;
- applying topological degree theory to boundary value problems involving rearrangements.

From suppositions on the measure space, such as the measure space having finite measure and no atoms, it can proved that the set of rearrangements is contractible and locally contractible. The Mountain Pass Lemma over rearrangements can be generalised, so instead of considering continuous paths from the closed unit interval to the set of rearrangements; it will consider the continuous functions from the closed unit disc into the set of rearrangements.

Topological degree theory is used to associate admissible triples of functions, sets and points with integers. These methods will be applied to a boundary value problem involving rearrangements, where the domain is almost equal to the union of balls, which has been studied using variational methods, providing new multiplicity results. The minimum number of solutions to this boundary value problem is found to be related exponentially to the number of balls contained in the domain.

1.2 An Introduction

The remainder of the opening chapter will define what rearrangements, or equimeasurable functions, are, and prove basic properties of rearrangements. This definition will be for finite measure spaces. Two functions being rearrangements are shown to be equivalent to six properties: pre-images of upper level sets have equal measures; preimages of Borel subsets of $\mathbb{R}$ have equal measures; the distribution functions are equal; the decreasing rearrangements are equal; the two functions have equal integrals when composed with non-negative Borel mesaurable functions; and the positive part of the functions minus arbitrary real values have equal integrals. In colloquial terms, two functions are rearrangements if they take the same values for the same amount. Thirteen properties of rearrangements are shown to hold. The chapter finishes with an historical overview and a literature review for rearrangements. If the reader is familiar with the study of rearrangements, then it should be possible to review this chapter quickly.

Chapter 2 focuses on the further study of rearrangements, and is split into two parts. The first section considers alternate characterisations of non-atomic and separable measure spaces, including measure resolutions. Measure resolutions of a measurable set are continuous functions of increasing subsets, from the unit interval beginning at negligible measurable subsets to sets that are equivalent – equal up to negligible measurable sets – to the given measurable set. The existence of a measure resolution of the whole measure space is shown to be equivalent to that measure space having no atoms. In the second subsection of this chapter, this characterisation of a finite and non-atomic measure space is used to prove that finite and non-atomic measure spaces have contractible and locally contractible sets of rearrangements. In the third subsection of this chapter, other standard rearrangement inequalities have assumed that the measure space is a *measure interval*: that is, having a measure-preserving bijection to a real interval equipped with the Lebesgue measure. These rearrangement inequalities will be shown to hold for weak assumptions on the underlying measure space.

Chapter 3 looks at the Mountain Pass Lemma over rearrangements, which is given in its original form in the Chapter 1 literature review. The lemmas given in preparation for the original version are all generalised, culminating with the main result. This result is then given an application, which is related to a boundary value problem involving rearrangements.

Chapter 4 discusses topological degree theory, from which I will be developing a method to find solutions to boundary value problems. In this chapter, topological degree theory will begin with an axiomatic treatment; before introducing the Leray-Schauder

Degree, for compact perturbations of the identity on infinite-dimensional real vector spaces. The chapter finishes with a short discussion on fixed point theorems, including Schauder's Fixed Point Theorem. Readers fluent with degree theory and fixed point theory can also swiftly review this chapter.

The fifth chapter develops the degree theory method for solving boundary value problem over rearrangements. These solutions are characterised in three distinct manners, including as the local maximisers of the energy functional. It is shown that these solutions can be characterised as the fixed points of a compact function from the weak closure of rearrangements into itself. Using that $L^p(\mu)$ is a uniformly convex Banach space for $1 < p < \infty$, a nearest point map is utilised to create a compact function from an open subset of $L^p(\mu)$ into that open subset. The fixed points of this compact function are then identified using the Leray-Schauder degree. The thesis considers a well-studied boundary value problem, where only variational methods had been used before: the negative Laplacian of a function may be written as itself composed with an increasing function that is unknown *a priori*, the function lies in the Sobolev space $H_0^1(\Omega)$ and the negative Laplacian of a function lies in a given set of rearrangements. The concern of this chapter is the multiplicity of solutions for the following equation:

$$\left.\begin{aligned} -\Delta\psi &= \varphi \circ \psi, \\ \psi &\in H_0^1(\Omega), \\ -\Delta\psi &\in \mathscr{F} \end{aligned}\right\}.$$

Here, I have used $\mathscr{F}$ to denote the set of rearrrangements of a prescribed, positive function. The function which describes the vorticity $-\Delta\psi$ must now lie in a set of rearrangements of a given function. The degree theory method is applied to this problem, where the domain Ω has a sufficiently smooth boundary and is approximately equal to the union of balls. The multiplicity results for the number of solutions to this problem is improved from estimates derived from variational methods, revealing that the minimum number of solutions is related exponentially to the number of balls contained in the domain Ω.

It should be considered how the two titular themes of my thesis are related. Rearrangements naturally arises out of fluid mechanics. If a particular subset of a body of fluid is focussed upon, then that fluid moves, the identified subset is rearranged over time. The rearrangements of sets can be used to define the rearrangements of functions, as seen in the historical overview in Chapter 1.

In fluid dynamics, the vorticity describes the local spinning motion of a fluid, seen from some observed point travelling along with the fluid. Mathematically, the vorticity of a three-dimensional flow is defined as the curl (or rotational) of the velocity field that describes the motion of the fluid. Note that the vorticity may be nonzero, even if all the particles are flowing along straight and parallel pathlines. This is the case for parallel flow with shear. A flow may also have zero vorticity even if all of its particles are flowing along curved trajectories, such as an irrotational vortex.

In three-dimensional Cartesian co-ordinates, it is formally defined as, where ∇ is the del operator:

$$\begin{aligned}\vec{\omega} &= \nabla \times \vec{v} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_x, v_y, v_z) \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \frac{\partial v_y}{\partial y} - \frac{\partial v_x}{-}\frac{\partial v_x}{\partial y}\right).\end{aligned}$$

In the two-dimensional case, the vorticity of a flow is perpendicular to the flow's plane, and so can be viewed as a scalar field:

$$\begin{aligned}\vec{\omega} &= \nabla \times \vec{v} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \times (v_x, v_y) \\ &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.\end{aligned}$$

The evolution of this vorticity through time is described by the *vorticity equation*. In [57, Section 2.1], the vorticity equation for an incompressible and isotropic fluid with conservative body forces simplies to the vorticity transport equation:

$$\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} = (\vec{\omega} \cdot \nabla)\vec{v} + \nu \Delta \vec{\omega},$$

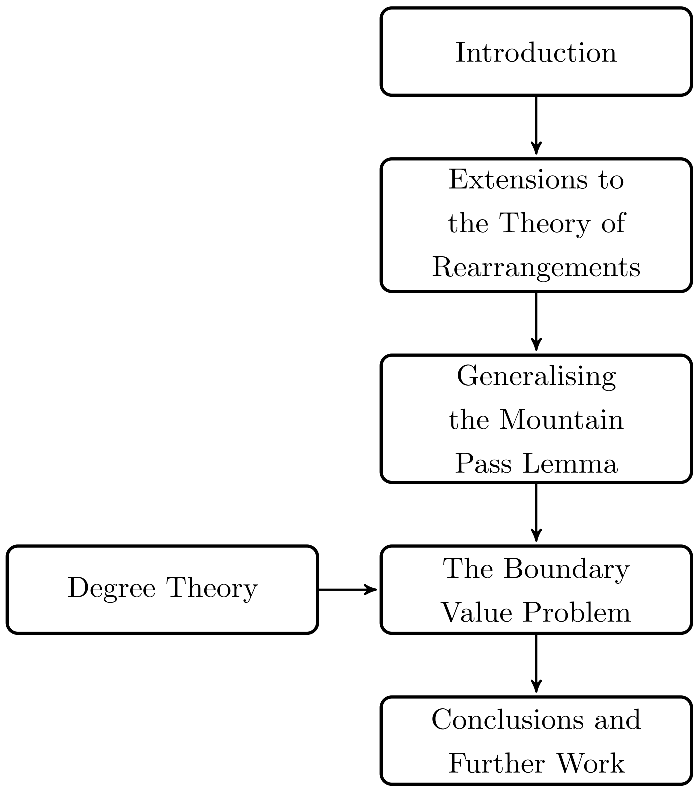
where ν is the kinematic viscocity and Δ is the Laplace operator.

For the two-dimensional case, since the vorticity is orthogonal to the kinematic viscocity in two dimensions, the vorticity-stretching term vanishes: $\omega \cdot \nabla \vec{v} \equiv 0$. This means the equation reduces to the *scalar vorticity equation*:

$$\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} = \nu \Delta \omega.$$

Chapter 5 will propose an alternative, via topological degree theory, to variational principles for stable steady vortices on isovortical surfaces, which is a set of flows whose vorticities are rearrangements of one another, as studied in [7]. This thesis will seek to build upon this existing corpus of work, providing a slick method by which a lower bound for the multiplicity of solutions to a well-known boundary value problem involving rearrangements can be calculated.

The following diagram details how the chapters link together:



The final chapter will consider what has been proven, how these results fit into the current corpus of work, and what future work might be undertaken. Broadly speaking, the Chapter 1 literature review, Chapter 2, Chapter 3 and Chapter 5 contains new work, whilst the beginning of Chapter 1 and the main section of Chapter 4 do not.

1.3 A Primer on Rearrangements

I begin by establishing basic definitions and notation. A *measure space* is a triple (X, Σ, μ), where X is a non-empty point set, Σ is a σ-algebra on X and μ is a measure on X. This means that $X \in \Sigma$, and Σ is closed under countable unions, and complements with respect to X, and μ satisfies that $0 \leq \mu(S) \leq \infty$ for every $S \in \Sigma$, and it is additive over countable disjoint subfamilies of Σ. A measure space is called *finite* if $\mu(X) < \infty$.

For brevity, I will often use X or (X, μ) instead of (X, Σ, μ) to represent a measure space. I now give the definitions of two functions being rearrangements, and the related concepts of the distribution function and the decreasing rearrangement of a function.

Definition 1.1 (Rearrangements)**.** Let (X, μ) and (Y, ν) be finite measure spaces. Suppose $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ are measurable functions. Then f and g are *rearrangements*, or *equimeasurable*, if

$$\mu\left(f^{-1}[\alpha, \infty)\right) = \nu\left(g^{-1}[\alpha, \infty)\right), \text{ for all } \alpha \in \mathbb{R}.$$

Remark 1.2. Two functions being rearrangements of one another forms an equivalence relation. A function having the same domain and being a rearrangement as a given function $f : X \to \mathbb{R}$ is also an equivalence relation. Given $f : X \to \mathbb{R}$, the *set of rearrangements of* f *on* X is denoted

$$\mathscr{R}_X(f) = \{g : X \to \mathbb{R} : \ g \text{ is a rearrangement of } f\},$$

and is equal to the equivalence class of $f : X \to \mathbb{R}$, where the functions have the domain X. It also follows from the definition that if there are two functions from measure spaces (X, μ) and (Y, ν) that are rearrangements, then $\mu(X) = \nu(Y)$.

It is possible to generalise the study of rearrangements to infinite measure spaces, as in [24]. In this thesis, I will only consider finite measure spaces.

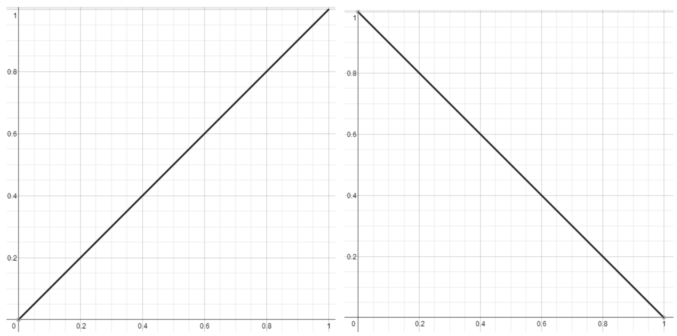
Example 1.3. To illustrate the meaning of Definition 1.1, I consider four functions $f_1, f_2, f_3, f_4 : [0, 1] \to [0, 1]$. These four functions take the following values:

$$\begin{aligned}
f_1(s) &= s; \\
f_2(s) &= 1 - s; \\
f_3(s) &= \begin{cases} s & \text{if } 0 \leq s \leq \frac{1}{2} \\ \frac{3}{2} - s & \text{if } \frac{1}{2} < s \leq 1. \end{cases}; \\
f_4(s) &= 2s \pmod 1.
\end{aligned}$$

These four functions are rearrangements of one another, so f_2, f_3 and f_4 all lie in the set of rearrangements of f_1 on $[0,1]$. In each case, for $i, j \in \{1,2,3,4\}$, it can be calculated that, for the Lebesgue measure μ_L on $[0,1]$,

$$\mu_L\left(\{s \in [0,1] : f_i(s) \geq \alpha\}\right) = 1 - \alpha = \mu_L\left(\{s \in [0,1] : f_j(s) \geq \alpha\}\right) \text{ for all } \alpha \in \mathbb{R}.$$

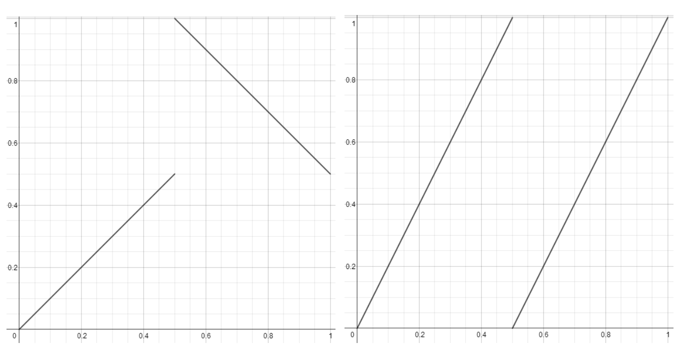
However, an alternative view of rearrangements may be derived from looking at the graphs of each function. The graph of f_2 is the graph of f_1 reflected in the vertical line $s = \frac{1}{2}$.



(a) f_1 (b) f_2

Figure 1-1: These are the graphs of the two functions $f_1, f_2 : [0,1] \to [0,1]$.

In the case of $f_3 : [0,1] \to [0,1]$, the graph of this function looks like the graph of the identity function, broken at the half-way point. The upper half of the graph is then reflected in the vertical line $s = \frac{3}{4}$. Lastly, $f_4 : [0,1] \to [0,1]$ is often called the doubling map.



(a) f_3 (b) f_4

Figure 1-2: These are the graphs fo the two functions $f_3, f_4 : [0,1] \to [0,1]$.

On intervals, rearrangements may be colloquially described as functions whose graphs may be rearranged into one another.

Definition 1.4 (Distribution function)**.** Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. The *distribution function* of f is defined by

$$d_f : \mathbb{R} \to [0, \mu(X)],\ d_f(t) = \mu\left(\{x \in X : f(x) > t\}\right).$$

It is immediate from the definition that $d_f : \mathbb{R} \to [0, \mu(X)]$ is a decreasing function.

Definition 1.5 (Decreasing rearrangement)**.** Let (X, Σ, μ) be a finite measure space, and suppose $f : X \to \mathbb{R}$ is measurable. Then $f^{\triangle} : [0, \mu(X)] \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the set of extended real numbers, is defined by $f^{\triangle}(0) = \operatorname{ess\,sup} f$ and for $t > 0$:

$$f^{\triangle}(t) = \inf\{\lambda \in \mathbb{R} : d_f(\lambda) \leq t\}.$$

The function $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is called the decreasing rearrangement of f.

Remark 1.6. If $f : X \to \mathbb{R}$, then in general, $f^{\triangle}(0) = \infty$. If $f \in L^{\infty}(X)$, then $f^{\triangle}(0) < \infty$.

Example 1.7. Suppose $f, g : [0,1] \to [0,1]$ take the values $f(s) = s$ and $g(s) = s^2$. By inspection, the distribution function of f has the domain $\mathbb{R}$ and range $[0,1]$, taking the values

$$d_f(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ 1 - s & \text{if } 0 < s < 1 \\ 0 & \text{if } s \geq 1 \end{cases}.$$

The decreasing rearrangment $f^{\triangle} : [0,1] \to [0,1]$ is simply $f^{\triangle}(s) = 1 - s$.

The distribution function of g is

$$d_g : \mathbb{R} \to [0,1],\ d_g(s) = \begin{cases} 1 & \text{if } s \leq 0 \\ 1 - \sqrt{s} & \text{if } 0 < s < 1 \\ 0 & \text{if} s \geq 1 \end{cases}.$$

For $0 < s < 1$, this may be calculated by:

$$\begin{aligned} d_g(s) &= \mu_L\left(\{x \in [0,1] : g(x) > s\}\right) \\ &= \mu_L\left(\left\{x \in [0,1] : x^2 > s\right\}\right) \\ &= \mu_L\left(\left\{x \in [0,1] : x > \sqrt{s}\right\}\right) \\ &= 1 - \sqrt{s}. \end{aligned}$$

The decreasing rearrangement $g^{\triangle} : [0,1] \to [0,1]$ is then $g^{\triangle}(s) = 1 - s^2$.

I now prove that the distribution function is right-continuous, and explain the relationship between the distribution function and the decreasing rearrangement of a given function.

Lemma 1.8. *Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. The distribution function $d_f : \mathbb{R} \to [0, \mu(X)]$ is decreasing and right-continuous.*

Proof. It is clear that d_f is decreasing. Denote $A_\lambda = \{x \in X : f(x) > \lambda\}$, and let $\lambda_0 \in \mathbb{R}$ be fixed. For all $n \in \mathbb{N}$, $\lambda_0 < \lambda_0 + \frac{1}{n}$, so it follows

$$A_{\lambda_0} = \bigcup_{n=1}^{\infty} A_{\lambda_0 + \frac{1}{n}}.$$

By monotonic convergence of the measure,

$$\lim_{n\to\infty} d_f\left(\lambda_0 + \frac{1}{n}\right) = \lim_{n\to\infty} \mu\left(A_{\lambda_0+\frac{1}{n}}\right) = \mu(A_{\lambda_0}) = d_f(\lambda_0).$$

The distribution function d_f is monotonically decreasing, so it is sufficient to consider $\left(\lambda_0 + \frac{1}{n}\right)_{n\in\mathbb{N}}$ to demonstrate right-continuity of d_f. □

Lemma 1.9. *Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Then $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is decreasing and right-continuous. Also,*

$$d_f(\lambda) > t \Longleftrightarrow f^{\triangle}(t) > \lambda, \text{ for all } t \in [0, \mu(X)] \text{ and } \lambda \in \mathbb{R}.$$

Proof. Let $r < s$, and then for every $t \in \mathbb{R}$,

$$\mu\left(\{x \in X : f(x) > t\}\right) \leq r \implies \mu\left(\{x \in X : f(x) > t\}\right) \leq s,$$

so it follows

$$\{t \in \mathbb{R} : d_f(t) \leq r\} \subseteq \{t \in \mathbb{R} : d_f(t) \leq s\}.$$

Taking infima on both sides yields $f^{\triangle}(r) \geq f^{\triangle}(s)$, so $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is decreasing. To show right-continuity of $f^{\triangle}$, taking any $y_0 \in (0, \mu(X))$ and let $(y_n)_{n\in\mathbb{N}}$ be such that $y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = y_0$. Then $x_n = f^{\triangle}(y_n) \leq x_0 = f^{\triangle}(y_0)$, so the sequence $(x_n)_{n\in\mathbb{N}}$ increases towards a limit $x \in \mathbb{R}$, which is bounded above by x_0. By the definition of the decreasing rearrangement, let $\varepsilon > 0$ be arbitrary, then

$$d_f(x_n - \varepsilon) > y_n \geq d_f(x_n + \varepsilon), \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

Assume, seeking contradiction, that $x < x_0$, then set $\varepsilon = \frac{1}{2}(x_0 - x)$, so

$$y_0 \geq d_f(x_0 + \varepsilon) \geq d_f(x_0 - \varepsilon) > y_0,$$

which is a contradiction. Hence, $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is right-continuous.

Since d_f is a decreasing function, it follows that

$$d_f(\lambda) > t \iff \lambda < \inf\{\alpha \in \mathbb{R} : d_f(\alpha) \leq t\} \iff f^{\triangle}(t) > \lambda,$$

as required. □

I now demonstrate that the naming system is internally consistent, that is, the decreasing rearrangement of a given function is a rearrangement of that function.

Proposition 1.10. *Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Then f and $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ are rearrangements.*

Proof. Let $s \in \mathbb{R}$ be arbitrary, then I want to find the distribution function of the decreasing rearrangement, denoting the Lebesgue measure on $[0, \mu(X)]$ by μ_L:

$$\begin{aligned}
d_{f^{\triangle}}(s) &= \mu_L\left(\left\{t \in [0, \mu(X)] : f^{\triangle}(t) > s\right\}\right) && \text{by definition} \\
&= \mu_L\left(\{t \in [0, \mu(X)] : d_f(s) > t\}\right) && \text{by Lemma 1.16} \\
&= \mu_L\left([0, d_f(s))\right) \\
&= d_f(s).
\end{aligned}$$

Hence, $d_{f^{\triangle}} = d_f$. For each $\alpha \in \mathbb{R}$, it follows

$$\mu\left(\{x \in X : f(x) > \alpha\}\right) = d_f(\alpha) = d_{f^{\triangle}}(\alpha) = \mu_L\left(\left\{t \in [0, \mu(X)] : f^{\triangle}(t) > \alpha\right\}\right).$$

Thus, f and $f^{\triangle}$ are rearrangements. □

I show that there are six different properties that are equivalent to two functions being rearrangements of one another.

Proposition 1.11. *Let (X, μ) and (Y, ν) be finite measure spaces with $\mu(X) = \nu(Y)$, and let $f : X \to \mathbb{R},\ g : Y \to \mathbb{R}$ be measurable functions. Then the following statements are equivalent:*

(i) f and g are rearrangements;

(ii) $\mu\left(f^{-1}(B)\right) = \nu\left(g^{-1}(B)\right)$ for every Borel set $B \subseteq \mathbb{R}$;

(iii) $d_f = d_g$;

(iv) $f^{\triangle} = g^{\triangle}$;

(v) $\int_X \varphi \circ f \ d\mu = \int_Y \varphi \circ g \ d\nu$ *for every non-negative Borel measurable function* $\varphi : \mathbb{R} \to \mathbb{R}$*;*

(vi) $\int_X (f-\alpha)_+ \ d\mu = \int_Y (g-\alpha)_+ \ d\nu$ *for all* $\alpha \in \mathbb{R}$*, where*

$$(f-\alpha)_+ : X \to \mathbb{R}, \ (f-\alpha)_+(x) = \max\{f(x)-\alpha, 0\}.$$

Proof. I use the following strategy for the proof. First, I prove that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$, then prove that $(ii) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (ii)$.

$\underline{(i) \Rightarrow (ii)}$ Suppose f and g are rearrangements, and denote the set of Borel sets of $\mathbb{R}$ by $\mathscr{B}(\mathbb{R})$. Define $f_*\mu, \ g_*\nu : \mathscr{B}(\mathbb{R}) \to [0,\infty]$ by

$$f_*\mu = \mu\left(f^{-1}(B)\right), \ g_*\nu = \nu\left(g^{-1}(B)\right), \text{ for all } B \in \mathscr{B}(\mathbb{R}).$$

Then $f_*\mu$ and $g_*\nu$ are both measures, and by supposition, agree on sets of the form $[\alpha,\infty), \ \alpha \in \mathbb{R}$, which generate the Borel sets of $\mathbb{R}$. It must be shown that the collection $\mathscr{D}$ of Borel sets where these measures agree form a σ-algebra. By supposition, if $A = [\alpha,\infty), \ B = [\beta,\infty) \in \mathscr{D}$ for some $\alpha, \beta \in \mathbb{R}$, then

$$A \cap B = [\max\{\alpha,\beta\},\infty) \in \mathscr{D}.$$

Hence, $\mathscr{D}$ is a π-system. Suppose $A, B \in \mathscr{D}$, where $A \subseteq B$ then

$$\begin{aligned}
f_*\mu(B \setminus A) &= \mu\left(f^{-1}(B \setminus A)\right) && \text{by definition}\\
&= \mu\left(f^{-1}(B)\right) - \mu\left(f^{-1}(A)\right) && \text{as } X \text{ has finite measure}\\
&= \nu\left(g^{-1}(B)\right) - \nu\left(g^{-1}(A)\right) && \text{by supposition}\\
&= \nu\left(g^{-1}(B \setminus A)\right) && \text{as } Y \text{ has finite measure}\\
&= g_*\nu(B \setminus A) && \text{by definition.}
\end{aligned}$$

Furthermore, suppose $(A_n)_{n\in\mathbb{N}}$ is a sequence that satisfies $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, in

$\mathscr{D}$. Then

$$\begin{aligned}
f_*\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right) && \text{by definition}\\
&= \lim_{m\to\infty} \mu\left(f^{-1}\left(\bigcup_{n=1}^{m} A_n\right)\right) && \text{by Monotone Convergence}\\
&= \lim_{m\to\infty} \mu\left(\bigcup_{n=1}^{m} f^{-1}(A_n)\right)\\
&= \lim_{m\to\infty} \nu\left(\bigcup_{n=1}^{m} g^{-1}(A_n)\right) && \text{by supposition}\\
&= \nu\left(g^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right)\right)\\
&= g_*\nu\left(\bigcup_{n=1}^{\infty} A_n\right).
\end{aligned}$$

Thus, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{D}$, and so $\mathscr{D}$ is a d-system.
By Dynkin's Lemma [71, Appendix A1.3], these two measures must agree on $\mathscr{B}(\mathbb{R})$, so for each Borel set $B \subseteq \mathbb{R}$, I observe

$$\mu\left(f^{-1}(B)\right) = f_*\mu(B) = g_*\nu(B) = \nu\left(g^{-1}(B)\right).$$

$\underline{(ii) \Rightarrow (iii)}$ Sets of the form $(t, \infty),\ t \in \mathbb{R}$ are Borel, so

$$d_f(t) = \mu(f^{-1}(t,\infty)) = \nu(g^{-1}(t,\infty)) = d_g(t).$$

$\underline{(iii) \Rightarrow (iv)}$ This is immediate, since

$$f^{\triangle}(t) = \inf\left\{\lambda \in \mathbb{R} : d_f(\lambda) < t\right\} = \inf\left\{\lambda \in \mathbb{R} : d_g(\lambda) < t\right\} = g^{\triangle}(t).$$

$\underline{(iv) \Rightarrow (i)}$ Suppose $f^{\triangle} = g^{\triangle}$, and recall two functions being rearrangements is an equivalence relation. By Proposition 1.14, f and $f^{\triangle}$, and g and $g^{\triangle}$ are two pairs of rearrangements, so by transitivity, f and g must be rearrangements.
$\underline{(ii) \Rightarrow (v)}$ First, I show this result holds with characteristic functions. Let $t \in \mathbb{R}$ and set $A = [t, \infty)$ with $\varphi = \mathbb{1}_A$, the characteristic function of A. By (ii),

$$\int_X \varphi \circ f \,\mathrm{d}\mu = \mu(f^{-1}(A)) = \nu(g^{-1}(A)) = \int_Y \varphi \circ g \,\mathrm{d}\nu.$$

Thus, the result holds for all characteristic functions. Simple functions are expressed as non-negative linear combinations of characteristic functions, so the result must also hold for all simple functions. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a non-negative Borel measurable function, then it is the limit of an increasing sequence of simple functions, $(\varphi_n)_{n\in\mathbb{N}}$. By the Monotone Convergence Theorem,

$$\int_X \varphi \circ f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X \varphi_n \circ f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_Y \varphi_n \circ g \, \mathrm{d}\nu = \int_Y \varphi \circ g \, \mathrm{d}\nu.$$

<u>*(v)* $\Rightarrow$ *(vi)*</u> For each $\alpha \in \mathbb{R}$, set $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$, $\varphi_\alpha(x) = \max\{x - \alpha, 0\}$. Each φ_α is a non-negative and continuous function, and so Borel measurable.
<u>*(vi)* $\Rightarrow$ *(ii)*</u> Here, I follow [31]. Fix $\alpha \in \mathbb{R}$, and then define $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(s) = \max\{s - \alpha, 0\}$. Then

$$\lim_{\tau\to 0^+} \frac{1}{\tau}[\varphi(s+\tau) - \varphi(s)] = \mathbb{1}_{(\alpha,\infty)}(s).$$

Hence, by Monotone Convergence,

$$\lim_{\tau\to 0^+} \frac{1}{\tau} \left\{ \int_X (f - \alpha - \tau)_+ \, \mathrm{d}\mu - \int_X (f-\alpha)_+ \, \mathrm{d}\mu \right\} = \mu\left(\{x \in X : f(x) > \alpha\}\right).$$

Since $\int_X (f-\beta)_+ \, \mathrm{d}\mu = \int_Y (g-\beta)_+ \, \mathrm{d}\nu$ for all $\beta \in \mathbb{R}$, we have

$$\mu\left(\{x \in X : f(x) > \alpha\}\right) = \nu\left(\{y \in Y : g(y) > \alpha\}\right).$$

Thus, $\mu(f^{-1}(B)) = \nu(g^{-1}(B))$ for all Borel sets $B \subseteq \mathbb{R}$. □

The immediate corollary of the Borel property, *(ii)* in Propsition 1.11, is that two functions have the same measure of pre-images of Borel generators are also rearrangements. This result is formulated below.

Corollary 1.12. *Suppose (X, Σ, ν) and (Y, Λ, ν) are two finite measure spaces, where $\mu(X) = \nu(Y)$. Suppose $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ are measurable functions. The functions f and g are rearrangements if, and only if, any one of the following conditions is met:*

(i) $\mu(\{x \in X : f(x) \geq \alpha\}) = \nu(\{y \in Y : g(y) \geq \alpha\})$ *for all* $\alpha \in \mathbb{R}$,

(ii) $\mu(\{x \in X : f(x) > \alpha\}) = \nu(\{y \in Y : g(y) > \alpha\})$ *for all* $\alpha \in \mathbb{R}$,

(iii) $\mu(\{x \in X : f(x) \leq \alpha\}) = \nu(\{y \in Y : g(y) \leq \alpha\})$ *for all* $\alpha \in \mathbb{R}$,

(iv) $\mu(\{x \in X : f(x) < \alpha\}) = \nu(\{y \in Y : g(y) < \alpha\})$ *for all* $\alpha \in \mathbb{R}$.

If any one of $(i) - (iv)$ *are satisfied, then*

$$\mu(\{x \in X : f(x) = \alpha\}) = \nu(\{y \in Y : g(y) = \alpha\}) \textit{ for all } \alpha \in \mathbb{R}.$$

Proof. The sets $\{[\alpha, \infty) : \alpha \in \mathbb{R}\}$, $\{(\alpha, \infty) : \alpha \in \mathbb{R}\}$, $\{(-\infty, \alpha] : \alpha \in \mathbb{R}\}$ and, lastly, $\{(-\infty, \alpha) : \alpha \in \mathbb{R}\}$ are all generators for the Borel sets of $\mathbb{R}$. By Proposition 1.11, if any of the conditions holds, then they all hold. For each $\alpha \in \mathbb{R}$,

$$\begin{aligned} \mu(\{x \in X : f(x) = \alpha\}) &= \mu(\{x \in X : f(x) \geq \alpha\}) - \mu(\{x \in X : f(x) > \alpha\}) \\ &= \nu(\{y \in Y : g(y) \geq \alpha\}) - \nu(\{y \in Y : g(y) > \alpha\}) \qquad \text{by } (i), (ii) \\ &= \nu(\{y \in Y : g(y) = \alpha\}). \end{aligned}$$

□

Remark 1.13. The converse statement – if

$$\mu(\{x \in X : f(x) = \alpha\}) = \nu(\{y \in Y : g(y) = \alpha\}) \text{ for all } \alpha \in \mathbb{R},$$

then f and g are rearrangements – is false. Set $X = Y = [0, 1]$, $f : [0, 1] \to [0, 1]$, $f(x) = x$ and $g : [0, 1] \to [0, 1]$, $g(x) = x^2$, and set μ and ν both be the Lebesgue measure μ_L on $[0, 1]$. It follows, for all $\alpha \in [0, 1]$:

$$\mu_L(\{x \in [0, 1] : x = \alpha\}) = \mu_L(\{\alpha\}) = 0 = \mu_L(\{\sqrt{\alpha}\}) = \mu_L(\{y \in [0, 1] : y^2 = \alpha\}).$$

However, f and g are plainly not rearrangements.

Next, I prove that a decreasing function on an interval $[0, \omega]$, where $\omega > 0$, is almost everywhere equal to its decreasing rearrangement. Consequently, the decreasing rearrangement on $[0, \omega]$ is essentially unique.

Lemma 1.14. *Let $f : [0, \omega] \to \mathbb{R}$ be decreasing, then $f = f^{\triangle}$ almost everywhere under the Lebesgue measure m on $[0, \omega]$.*

Proof. Suppose $t < f(s)$, then $\mu_L(\{y \in [0, \omega] : f(y) > t\}) \geq s$. Thus, by definition of the decreasing rearrangement, $f^{\triangle}(s) \geq t$. Since this t was arbitrary, it follows $f^{\triangle}(s) \geq f(s)$ for all $s \in [0, \omega]$. Now, let $s \in (0, \omega)$ be a point of continuity of f. Since f is non-increasing, we have for $\delta > 0$,

$$\mu_L(\{y \in [0, \omega] : f(y) > f(s - \delta)\}) \leq s - \delta < s$$

which implies $f^{\triangle}(s) \leq f(s - \delta)$. Applying the continuity at $s \in (0, \omega)$ and the limit $\delta \to 0^+$ yields $f^{\triangle}(s) \leq f(s)$. Monotone functions on an interval have only a countable number of discontinuities, and all countable sets have Lebesgue measure zero, so $f = f^{\triangle}$ almost everywhere. □

Corollary 1.15. *Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Then $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is a right-continuous, decreasing rearrangement of f, and if $g : [0, \mu(X)] \to \mathbb{R}$ is a right-continuous, decreasing rearrangement of f, then $g = f^{\triangle}$ up to sets of Lebesgue measure zero.*

Proof. This is a consequence of the above lemma, since $g = g^{\triangle} = f^{\triangle}$ up to sets of Lebesgue measure zero. $\square$

Now, I will show the map which sends every function to its distribution function sends pointwise limits to pointwise limits. This result will be required to prove properties about rearrangements.

Lemma 1.16. *Let (X, Σ, μ) be a finite measure space, and let $f_n, f : X \to \mathbb{R}$ be measurable functions for all $n \in \mathbb{N}$. Suppose $\lim_{n\to\infty} f_n = f$ pointwise almost everywhere, then $\lim_{n\to\infty} d_{f_n} = d_f$ at every point of continuity of d_f, which is almost everywhere.*

Proof. Monotone functions have negligible sets of discontinuity, so $d_f : \mathbb{R} \to [0, \mu(X)]$ is continuous almost everywhere. Suppose d_f is continuous at $t \in \mathbb{R}$. Let $\varepsilon > 0$ be arbitrary, then by the continuity of d_f at $t \in \mathbb{R}$, there exists $\delta > 0$ such that for all $s \in (t - 2\delta, t + 2\delta)$ implies $|d_f(s) - d_f(t)| < \frac{1}{2}\varepsilon$. Then, by the monotonicity of the measure and the Triangle Inequality,

$$
\begin{aligned}
\mu(\{x \in X : f(x) = t\}) &\leq \mu(\{x \in X : t - \delta < f(x) < t + \delta\}) \\
&= |d_f(t+\delta) - d_f(t-\delta)| + |d_f(t) - d_f(t-\delta)| \\
&< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, $\mu(\{x \in X : f(x) = t\}) = 0$. As $\lim_{n\to\infty} f_n = f$ pointwise almost everywhere, set $S = \{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}$ and $\mu(S) = 0$. For measurable sets $A_n \subset \Sigma,\ n \in \mathbb{N}$ with finite measure, the notation I will be using is:

$$
\begin{aligned}
\liminf\,(A_n) &= \bigcup_{n=1}^{\infty} \left(\bigcap_{m=n}^{\infty} A_m \right); \\
\limsup\,(A_n) &= \bigcap_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m \right).
\end{aligned}
$$

Suppose $t \in \mathbb{R}$ is a point of contuinity of d_f, then

$$
\begin{aligned}
\{x \in X : f(x) > t\} &\subseteq \liminf\,\{x \in X : f_n(x) > t\} && \text{by pointwise convergence} \\
&\subseteq \limsup\,\{x \in X : f_n(x) > t\} && \text{by definition of } \limsup \\
&\subseteq \{x \in X \setminus S : f(x) \geq t\} \cup S && \text{by inertia.}
\end{aligned}
$$

For any sequence of measurable sets with finite measure $(A_n)_{n\in\mathbb{N}} \subset \Sigma$, it is known, such as in [13, Theorem 2.21], that

$$\mu(\liminf A_n) \le \liminf \mu(A_n) \le \limsup \mu(A_n) \le \mu(\limsup A_n).$$

Applying this result to $A_n = \{x \in X : f_n(x) > t\}$ yields

$$\begin{aligned} d_f(t) &= \mu(\{x \in X : f(x) > t\}) \\ &\le \liminf \mu(\{x \in X : f_n(x) > t\}) = \liminf d_{f_n}(t) \\ &\le \limsup \mu(\{x \in X : f_n(x) > t\}) = \limsup d_{f_n}(t) \\ &\le \mu(\{x \in X : f(x) > t\}) + \mu(\{x \in X : f(x) = t\}) + \mu(S) \\ &\le \mu(\{x \in X : f(x) > t\}) = d_f(t). \end{aligned}$$

The latter two inequalities hold due to prior calculation, and since $f^{-1}(\{t\})$ and S are negligible.

Thus, $\lim_{n\to\infty} d_{f_n} = d_f$ pointwise almost everywhere, as required. □

Denote the positive and negative parts of a measurable function $f : X \to \mathbb{R}$ by $f_+ : X \to \mathbb{R}$, $f_+(x) = \max\{f(x), 0\}$ and $f_- : X \to \mathbb{R}$, $f_-(x) = -\min\{-f(x), 0\}$ respectively, so that $f = f_+ - f_-$ and $|f| = f_+ + f_-$. Also, the concept of a measure-preserving map will be defined, and an alternate characterisation is given.

Definition 1.17 (Measure-Preserving Map, Measure-Preserving Bijection)**.** Let (X, Σ, μ) and (Y, Λ, ν) be two finite measure spaces. $\sigma : Y \to X$ is a *measure-preserving map* if for all $A \in \Sigma$,

$$\sigma^{-1}(A) \in \Lambda \text{ and } \nu(\sigma^{-1}(A)) = \mu(A).$$

If $\sigma : Y \to X$ is invertible, and both σ and σ^{-1} are measure-preserving maps, then σ is a *measure-preserving bijection.*

Lemma 1.18. *Let (X, Σ, μ) be a measure space. A measurable transformation $\sigma : X \to X$ is measure-preserving with respect to μ if, and only if, for all μ-integrable functions $f : X \to \mathbb{R}$,*

$$\int_X f \ d\mu = \int_X (f \circ \sigma) \ d\mu.$$

Proof. (Sufficiency) Suppose the integral property holds. Let $A \in \Sigma$ be a measurable set, then setting $f = \mathbb{1}_A$:

$$\mu(A) = \int_X \mathbb{1}_A \ \mathrm{d}\mu = \int_X (\mathbb{1}_A \circ \sigma) \ \mathrm{d}\mu = \int_X \mathbb{1}_{\sigma^{-1}(A)} \ \mathrm{d}\mu = \mu(\sigma^{-1}(A)).$$

Hence, σ is measure-preserving.
(Necessity) Suppose σ is measure-preserving, then the integral property holds for all indicator functions. By the linearity of the integral, it holds for all simple functions. Suppose $f \geq 0$ and is integrable, and $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence of simple functions with limit f. Then $\lim_{n\to\infty} f_n \circ \sigma = f \circ \sigma$. By Lebesgue's Dominated Convergence Theorem, it follows that

$$\int_X (f \circ \sigma)\ \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \circ \sigma\ \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n\ \mathrm{d}\mu = \int_X f\ \mathrm{d}\mu.$$

Let $f : X \to \mathbb{R}$ be an arbitrary integrable function over X, then the decomposition $f = f_+ - f_-$ and the result for non-negative functions yields the final result. □

Definition 1.19 (Essential range)**.** Let (X, Σ, μ) be a finite measure space, and let $f : X \to \mathbb{R}$ be a measurable function. The *essential range* of f is

$$\{\alpha \in \mathbb{R} : \mu\left(\{x \in X : \alpha - \varepsilon < f(x) < \alpha + \varepsilon\}\right) > 0 \text{ for all } \varepsilon > 0\}.$$

The following result is a compendium of basic properties about rearrangements, which will be utilised throughout the thesis.

Proposition 1.20. *Let (X, Σ, μ) and (Y, Λ, ν) be two finite measure spaces such that $\mu(X) = \nu(Y)$. Denote that measurable functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are rearrangements by $f \sim g$. Suppose $f,\ f',\ f_n : X \to \mathbb{R}$ and $g,\ g',\ g_n : Y \to \mathbb{R}$ are measurable functions for all $n \in \mathbb{N}$.*

(i) *Suppose $f \sim g$, then f and g have the same essential range,*

(ii) *Let f be a simple function, and $g \sim f$, then g is a simple function,*

(iii) *If $f \sim g$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function, then $\varphi \circ f \sim \varphi \circ g$,*

(iv) *Suppose $f \sim g$, then for every $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta \sim \alpha g + \beta$,*

(v) *If $f \sim g$, then $|f| \sim |g|$,*

(vi) *Let $\sigma : Y \to X$ be a measure-preserving map, then $f \sim f \circ \sigma$,*

(vii) *Suppose $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ converge pointwise almost everywhere, and $f_n \sim g_n$ for all $n \in \mathbb{N}$, then $f \sim g$,*

(viii) *Suppose $1 \leq p \leq \infty$, and let $f_n,\ f \in L^p(X)$ and $g_n,\ g \in L^p(Y)$, and $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$ in their respective p-norms. If $f_n \sim g_n$ for all $n \in \mathbb{N}$, then $f \sim g$,*

(ix) *Suppose $f, g \geq 0$ and $f \sim g$, then $\int_X f \; d\mu = \int_Y g \; d\nu$, either being both infinite or both are finite and take the same value,*

(x) *Suppose $f \sim g$ and $f \in L^1(X)$ and $g \in L^1(Y)$, then $\int_X f d\mu = \int_Y g \; d\nu$,*

(xi) *If $1 \leq p \leq \infty$ and $f \in L^p(X)$ and $g \sim f$, then $g \in L^p(Y)$ and $||f||_p = ||g||_p$,*

(xii) *Let $A_n \in \Sigma$ for all $n \in \mathbb{N}$ is a partition of X, that is, $X = \bigcup_{n=1}^{\infty} A_n$ and $A_i \cap A_j =$ for all $i, j \in \mathbb{N}$ where $i \neq j$. Suppose $f|_{A_n} \sim f'|_{A_n}$ for all $n \in \mathbb{N}$, then $f \sim f'$.*

(xiii) *Suppose $A \in \Sigma$, and if $f|_A \sim f'|_A$ and $f \sim f'$, then $f|_{X \setminus A} \sim f'|_{X \setminus A}$.*

Proof. (i) Let $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. From Corollary 1.12, it follows that

$$\mu(f^{-1}(\alpha - \varepsilon, \alpha + \varepsilon)) = \nu(g^{-1}(\alpha - \varepsilon, \alpha + \varepsilon)).$$

Hence, the essential range of f and g must be equal.

(ii) Recall that a function is simple if, and only if, it is non-negative and has finite range. Suppose f is a simple function and $g \sim f$, then g is also non-negative and has the same essential range of f, by (i), so the range of g is finite too. It follows that g is simple.

(iii) Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel-measurable function. Let $B \subseteq \mathbb{R}$ be a Borel set, then $\varphi^{-1}(B)$ is also Borel. Then, since $f \sim g$,

$$\mu((\varphi \circ f)^{-1}(B)) = \mu(f^{-1}\varphi^{-1}(B)) = \nu(g^{-1}\varphi^{-1}(B)) = \nu((\varphi \circ g)^{-1}(B)),$$

so $\varphi \circ f \sim \varphi \circ g$.

(iv) Set $\varphi : \mathbb{R} \to \mathbb{R}, \; \varphi(x) = \alpha x + \beta$ in (iii), which is a continuous function, and so Borel-measurable.

(v) Set $\varphi : \mathbb{R} \to \mathbb{R}, \; \varphi(x) = |x|$, which is continuous.

(vi) Let $\alpha \in \mathbb{R}$ be arbitrary, then

$$\nu\left((f \circ \sigma)^{-1}[\alpha, \infty)\right) = \nu\left(\sigma^{-1} f^{-1}[\alpha, \infty)\right) = \mu\left(f^{-1}[\alpha, \infty)\right),$$

so it follows that $f \sim f \circ \sigma$.

(vii) By Proposition 1.11, it is known that $f_n \sim g_n$ for all $n \in \mathbb{N}$ if, and only if, $d_{f_n} = d_{g_n}$ for all $n \in \mathbb{N}$. By Lemma 1.16,

$$d_f = \lim_{n \to \infty} d_{f_n} = \lim_{n \to \infty} d_{g_n} = d_g.$$

Another application of Proposition 1.11 yields $f \sim g$.

$(viii)$ Convergence in the p-norm implies the existence of a subsequence that converges pointwise. Relabel these subsequences as $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$, then apply (vii).

(ix) Firstly, I prove this is true for characteristic functions. Let $A \in \Sigma$ and $B \in \Lambda$, then $\mathbb{1}_A \sim \mathbb{1}_B$ if, and only if, $\mu(A) = \nu(B)$. Considering the integrals of these characteristic functions yields,

$$\int_X \mathbb{1}_A \, \mathrm{d}\mu = \mu(A) = \nu(B) = \int_Y \mathbb{1}_B \, \mathrm{d}\nu.$$

By linearity of the integral, this holds for all simple functions. Suppose $f, g \geq 0$ and $f \sim g$, then there exists two increasing sequences of simple functions such that $f_n \sim g_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$. By this result for simple functions,

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int_Y g_n \, \mathrm{d}\nu = \int_Y g \, \mathrm{d}\nu.$$

(x) Suppose $f \sim g$ with $f \in L^1(X)$ and $g \in L^1(Y)$. Decompose the two functions into their positive and negative parts, so $f = f_+ - f_-$ and $g = g_+ - g_-$. By applying (iii) with $\varphi : \mathbb{R} \to \mathbb{R},\ \varphi(x) = \max\{x, 0\}$ and $\varphi' : \mathbb{R} \to \mathbb{R},\ \varphi'(x) = -\min\{-x, 0\}$, it is yielded that $f_+ \sim g_+$ and $f_- \sim g_-$. Applying (ix) and the functions lie in $f \in L^1(X)$ and $g \in L^1(Y)$:

$$\int_X f \, \mathrm{d}\mu = \int_X f_+ \, \mathrm{d}\mu - \int_X f_- \, \mathrm{d}\mu = \int_Y g_+ \, \mathrm{d}\nu - \int_Y g_- \, \mathrm{d}\nu = \int_Y g \, \mathrm{d}\nu.$$

(xi) For $1 \leq p < \infty$, use (iii) with $\varphi : \mathbb{R} \to \mathbb{R},\ \varphi(x) = |x|^p$, which is continuous and therefore Borel-measurable. Hence, if $f \in L^p(X)$, then $|f|^p \sim |g|^p$ by (iii). Note that $|f|^p \in L^1(X)$, and so by (ix), it is found that $\int_X |f|^p \, \mathrm{d}\mu = \int_Y |g|^p \, \mathrm{d}\nu < \infty$. Hence, $g \in L^p(Y)$. From this calculation,

$$\int_X |f|^p \, \mathrm{d}\mu = \int_Y |g|^p \, \mathrm{d}\nu \text{ implies that } ||f||_p = ||g||_p.$$

If $f \in L^\infty(X)$ and $f \sim g$, assume, seeking contradiction, that $\operatorname{ess\,sup}_Y g > \operatorname{ess\,sup}_X f$. Let $\alpha \in \mathbb{R}$ satisfy $\operatorname{ess\,sup}_Y g > \alpha > \operatorname{ess\,sup}_X f$, then

$$\nu(Y) = \mu(X) = \mu(|f|^{-1}[\alpha, \infty)) = \nu(|g|^{-1}[\alpha, \infty)) < \nu(Y)$$

which is a contradiction. Hence, $\operatorname{ess\,sup}_X f = \operatorname{ess\,sup}_Y g$, so $f \in L^\infty(X)$ if, and only if, $g \in L^\infty(Y)$, given that $f \sim g$.

(xii) Suppose $\{A_n : n \in \mathbb{N}\}$ is a measurable partition of X. Then, for each $\alpha \in \mathbb{R}$,

$$\begin{aligned} \mu(\{x \in X : f(x) \geq \alpha\} &= \sum_{n=1}^{\infty} \mu(\{x \in X : f(x) \geq \alpha\} \cap A_n) && \text{by partition} \\ &= \sum_{n=1}^{\infty} \mu(\{x \in A_n : f|_{A_n}(x) \geq \alpha\}) \\ &= \sum_{n=1}^{\infty} \mu(\left\{x \in A_n : f'|_{A_n}(x) \geq \alpha\right\}) && \text{by supposition} \\ &= \sum_{n=1}^{\infty} \mu(\left\{x \in X : f'(x) \geq \alpha\right\} \cap A_n) \\ &= \mu\left\{x \in X : f'(x) \geq \alpha\right\} && \text{by partition.} \end{aligned}$$

Alternately, setting $B_m = \bigcup_{n=1}^{m} A_n$ for each $m \in \mathbb{N}$, and $f_m = f|_{B_m}$ and $f'_m = f'|_{B_m}$, means an application of (vii) yields the result.

$(xiii)$ Let $\alpha \in \mathbb{R}$ be arbitrary. Then, by the measure axioms and the supposition that $f \sim f'$,

$$\begin{aligned} \mu\left(\{x \in X \setminus A : f(x) \geq \alpha\}\right) &= \mu\left(\{x \in X : f(x) \geq \alpha\}\right) - \mu\left(\{x \in A : f(x) \geq \alpha\}\right) \\ &= \mu\left(\left\{x \in X : f'(x) \geq \alpha\right\}\right) - \mu\left(\left\{x \in A : f'(x) \geq \alpha\right\}\right) \\ &= \mu\left(\left\{x \in X \setminus A : f'(x) \geq \alpha\right\}\right). \end{aligned}$$

By Definition 1.1, $f|_{X \setminus A} \sim f'|_{X \setminus A}$.

□

1.4 Historical Overiew and Literature Review

John V. Ryff wrote several papers on the subjects of rearrangements, including [65, 66, 64, 67].

In [65], Ryff seeks to generalise a partial order introduced in the classical treatise on inequalities [40]. Hardy, Littlewood and Pólya defined the most basic form of rearrangements, which is the rearrangement of a finite ordered list of numbers. Denote $\mathbb{R}^N$ to be the N-dimensional real space, and let $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$. Set $x^* = (x^*_1, \ldots, x^*_N)$ and $y^* = (y^*_1, \ldots, y^*_N)$ be vectors in $\mathbb{R}^N$ obtained by rearranging the co-ordinates of x and y into descending order. Then define $y \prec x$ if, and only, if

$$\sum_{i=1}^{n} y_i^* \leq \sum_{i=1}^{n} x_i^* \text{ for all } 1 \leq n \leq N,$$
$$\sum_{i=1}^{N} y_i = \sum_{i=1}^{N} x_i.$$

Rearrangements of vectors are simply permutation functions, usually denoted as elements of S_N, applied to the co-ordinates of the vectors. Under this partial order, it may be said that x and y are rearrangements if, and only if, $y \prec x$ and $x \prec y$. It is clear that x^* is a rearrangement of x, and both $x \prec x^*$ and $x^* \prec x$. Hardy, Littlewood and Pólya also defined the rearrangement of a function, limited to "functions of a continuous variable". The authors had supposed that $\phi(x)$ was non-negative and integrable over $(0, 1)$, but had identified the decreasing rearrangement of a function on $(0, 1)$, which has "in general, an infinite peak at the origin". In the original paper, it was denoted $\overline{\phi}$. To conform to my notation, I have relabelled this function as $\phi^{\triangle}$.

Despite their inclusion of superfluous suppositions, and working from the finite-dimensional vector space analogues, Hardy, Littlewood and Pólya had included a proof of one of the most famous rearrangement inequalities:

Theorem 1.21. *Let $\omega > 0$. Suppose $\phi, \psi : (0, \omega) \to [0, \infty)$ are two bounded and measurable functions. Whether $\omega < \infty$ or $\omega = \infty$,*

$$\int_0^{\omega} \phi(x)\psi(x)\ \mathrm{d}x \leq \int_0^{\omega} \phi^{\triangle}(s)\psi^{\triangle}(s)\ \mathrm{d}s.$$

Proof. This is shown in [40, Theorem 378 (p. 278)]. □

The partial ordering set out in Chapter X of [40] was generalised to $L^1(0, 1)$ by Ryff. Suppose $f : (0, 1) \to \mathbb{R}$ is a measurable function, then denote its decreasing rearrangement by $f^{\triangle} : (0, 1) \to \mathbb{R}$. If $f, g \in L^1(0, 1)$, then $g \prec f$ if, and only if,

$$\int_0^{s} g^{\triangle}(x)\ \mathrm{d}x \leq \int_0^{s} f^{\triangle}(x)\ \mathrm{d}x \text{ for all } 0 \leq s \leq 1,$$
$$\int_0^{1} g(x)\ \mathrm{d}x = \int_0^{1} f(x)\ \mathrm{d}x.$$

The papers [66] and [65] further investigate the partial order $\prec$ on $L^1(0, 1)$, considering the properties on the set $\mathrm{Orb}(f) = \{g \in L^1(0, 1) : g \prec f\}$. In [66], this set is referred to as the *orbit* of f; as the set of doubly stochastic operators, to which every $g \prec f$ must have a doubly stochastic operator T such that $g = Tf$, forms a semigroup. A *doubly* *stochastic* operator $T : L^1(X) \to L^1(X)$ is a linear operator such that $Tf \prec f$ for all $f \in L^1(X)$. An exposed point of a convex set C is an extreme point through which a closed supporting hyperplane may be passsed containing no other points of C. Ryff shows that every extreme point of $\mathrm{Orb}(f)$ is also an exposed point. Other properties of orbits are considered, such as the extreme points of each orbit are closed in the norm topology, which is clear once it is established for measurable functions that $f_n \to f$ implies $f_n^{\triangle} \to f^{\triangle}$. Also, the set

$$\{f^* \circ \sigma : \sigma \text{ is invertible and measure-preserving}\}$$

is norm-dense in the extreme points of $\Omega(f)$, where a measure-preserving transformation σ is said to be *invertible* if it is essentially univalent with σ^{-1} measure-preserving. In [65], the following characterisation of the extreme points $\mathrm{Extr}(f)$ of $\Omega(f)$ is offered:

$$\mathrm{Extr}(f) = \left\{f^{\triangle} \circ \sigma : \sigma \text{ is measure-preserving}\right\}.$$

Now, for each $f \in L^1(0,1)$, $\mathrm{Extr}(f)$ is weakly dense in $\mathrm{Orb}(f)$.

In [67], Ryff seeks to scrutinise the relationship between a function and its decreasing rearrangement. First, Ryff creates a measure-preserving transformation that takes f to $f^{\triangle}$, amalgamating the results from [67, Proposition 1 (p. 450)], [67, Proposition 2 (p. 451)] and [67, Proposition 3 (p. 452)]:

Proposition 1.22. *If $f : (0,1) \to \mathbb{R}$ is measurable, define*

$$\sigma(x) = \mu\left(\{y \in (0,1) : f(y) > f(x)\}\right) + \mu\left(\{y \in (0,1) : f(y) = f(x),\ y \leq x\}\right),$$

then $\sigma : (0,1) \to (0,1)$ is measurable. It is also measure-preserving and

$$f = f^{\triangle} \circ \sigma.$$

Ryff then compares the derivatives of f and $f^{\triangle}$, in the case where both of these functions have derivatives [67, Theorem (p. 455)]:

Theorem 1.23. *If f is differentiable almost everywhere, then for any $p > 0$:*

$$\int_0^1 |(f^{\triangle})'(x)|^p \,\mathrm{d}x \leq \int_0^1 |f'(x)|^p \,\mathrm{d}x.$$

The direct corollaries of this result, found at [67, Corollary (p. 456)] are that the decreasing rarrangement of a singular function is again singular and there are no singular measure-preserving transformations. Here, a *singular* function is one that has a derivative that vanishes almost everywhere. Lastly, it is shown, in [67, Theorem (p. 457)] that if f is absolutely continuous, then so is $f^{\triangle}$.

There are three streams of work in the study of rearrangements: theoretical generalisation, applications to physical problems and the study of Lorentz spaces and interpolation spaces. Since the thesis will focus on the first two of these streams, the study of Lorentz spaces and interpolation should be considered here, as described in [8]. The Lorentz space on a finite measure space (X, Σ, μ) is the space of all complex-valued measurable functions $f : X \to \mathbb{C}$, where the following quasinorm is finite:

$$\|f\|_{L^{p,q}(X)} = p^{\frac{1}{q}} \left\| t \left(\mu \left(\{ x \in X : |f(x)| \geq t \} \right) \right)^{\frac{1}{p}} \right\|_{L^q\left(\mathbb{R}, \frac{dt}{t}\right)}$$

where $0 < p < \infty$ and $0 < q \leq \infty$. In the case where $q < \infty$, the Lorentz quasinorm is

$$\|f\|_{L^{p,q}(X)} = p^{\frac{1}{q}} \left(\int_0^\infty t^q \mu \left(\{ x \in X : |f(x)| \geq t \} \right)^{\frac{q}{p}} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}.$$

When $q = \infty$, the quasinorm is

$$\|f\|_{L^{p,\infty}(X)} = \sup_{t>0} \left(t^p \mu \left(\{ x \in X : |f(x)| > t \} \right) \right).$$

The concept of rearrangements may be used to simplify these formulae. For a complex-valued measurable function $f : X \to \mathbb{C}$, the decreasing rearrangement f^* may be defined as

$$f^* : [0, \infty) \to [0, \infty], \;\; f^*(t) = \inf \left\{ \alpha \in \mathbb{R}^+ : d_f(\alpha) \leq t \right\}$$

for a distribution function d_f of f, with the descriptive formula:

$$d_f(\alpha) = \mu \left(\{ x \in X : |f(x)| > \alpha \} \right).$$

The infimum of the empty set is defined by ∞. Thus, $|f|$ and f^* are equimeasurable (and hence rearrangements according to Definition 1.1). Given this definition of $f^* : [0, \infty) \to [0, \infty]$, the Lorentz quasinorm becomes

$$\|f\|_{L^{p,q}(X)} = \begin{cases} \left(\int_0^\infty \left(t^1 p f^*(t) \right)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} & \text{if } q \in (0, \infty), \\ \sup_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty. \end{cases}$$

By definition, the Lorentz quasinorm is invariant between equimeasurable functions. This notion of invariance with respect to rearrangements may be generalised. First, I must introduce the concept of a Banach function norm, which is given in [8, Definition

1.1].

Definition 1.24 (Banach function norm)**.** Suppose (X, Σ, μ) is a measure space. Let $\mathscr{M}^+$ denote the cone of μ-measurable functions on X with values in $[0, \infty]$. A mapping $\rho : \mathscr{M}^+ \to [0, \infty]$ is called a *Banach function norm* if, for all $f, g, f_n \in \mathscr{M}$ with $n \in \mathbb{N}$ and $\alpha \geq 0$, and all $E \in \Sigma$, the following properties hold:

(P1) $\rho(f) = 0 \;\Leftrightarrow\; f = 0$ μ-a.e; $\rho(\alpha f) = \alpha\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ μ-a.e implies that $\rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{R}$ and $\lim_{n\to\infty} f_n = f$ μ-a.e., then $\lim_{n\to\infty} \rho(f_n) = \rho(f)$;

(P4) $\mu(E) < \infty$ implies that $\rho(\mathbb{1}_E) < \infty$;

(P5) $\mu(E) < \infty$ implies that $\int_E f \,\mathrm{d}\mu \leq C_E\rho(f)$, where $0 < C_E < \infty$ dependent on E and ρ, but it is independent of f.

Definition 1.25 (Rearrangement-Invariant Space)**.** Let ρ be a function norm over a finite measure space (X, Σ, μ), and let $\mathscr{M}_0^+(X)$ denote the set of all functions that non-negative real numbers and are finite μ-almost everywhere. Then ρ is called *rearrangement-invariant* if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions $f, g \in \mathscr{M}_0^+(X)$. The Banach function space $(Y, \|\cdot\|_Y)$ generated by ρ, where $\|f\|_Y = \rho(|f|)$, is said to be a *rearrangement-invariant space.*

Example 1.26. L^p spaces, for $1 \leq p \leq \infty$, are rearrangement-invariant Banach function spaces. Indeed, the sum $L^1 + L^\infty$ and the intersection $L^1 \cap L^\infty$ of L^1 and L^∞ are the largest and the smallest rearrangement-invariant Banach function spaces. For this reason, these spaces become crucial in the study of the interpolation of operators, considered in [8].

The most basic form of the study of rearrangements is the rearrangement of real vectors. The next level of generalisation is the rearrangement of real sequences, which was investigated by Albayrak and Pehlivan in [3]; and the rearrangements of series of real vectors, as in [55] by Nash-Williams and White. In [4], Almgren and Lieb define rearrangements as elements of Sobolev spaces, using the co-area formula to define symmetric-decreasing rearrangements. Their definition of a symmetric-decreasing rearrangement of nonnegative-valued functions from $\mathbb{R}^N$ is as follows: let $\alpha(N)$ denote the volume of a ball of radius one in $\mathbb{R}^N$ and for each $y > 0$, define the radius $R(y) = R_f(y)$ by

$$\alpha(N)R(y)^N = \int_{\mathbb{R}^N} \mathbb{1}_{\{x\in\mathbb{R}^N : f(x)>y\}} \,\mathrm{d}\mu = \mathscr{V}_f(y).$$

Let $\mathscr{C}_0$ denote the set of functions where

$$\mu\left(\left\{x \in \mathbb{R}^N : f(x) > y\right\}\right) < \infty \text{ for each positive number } y > 0,$$

then clearly $R(y) < \infty$ for $f \in \mathscr{C}_0$. Let $\mathbb{1}_R$ denote the characteristic function of the open ball of R centred at $0 \in \mathbb{R}^N$. Then

$$f^{\triangle}(x) = \int_0^{\infty} \mathbb{1}_{R(y)}(x) \ \mathrm{d}\mu(y).$$

This formula is usually called the *layer-cake representation of a function.* This definition is similar to the one given in [51], where Lieb collaborated with Loss. Lieb and Loss also wrote a general rearrangement inequality [51, Theorem 3.8], originally in [9]:

Theorem 1.27 (Generalised Riesz Rearrangement Inequality)**.** *Let $f_1, \ldots, f_n$ be non-negative functions on $\mathbb{R}^N$, vanishing at infinity. Let $m \leq n$, and let $B = (b_{ij})$ be a $m \times n$ matrix, with $1 \leq i \leq m,\ 1 \leq j \leq n$. Define*

$$I(f_1, \ldots, f_n) := \int_{\mathbb{R}^N} \cdots \int_{\mathbb{R}^N} \prod_{j=1}^{n} f_j \left(\sum_{i=1}^{m} b_{ij} x_i \right) dx_1 \cdots dx_m.$$

Then

$$I(f_1, \ldots, f_n) \leq I(f_1^{\triangle}, \ldots, f_n^{\triangle}).$$

Draghici also proved a similar rearrangement inequality in [26]. When the variables in this inequality are set to

$$n = 3,\ m = 2 \text{ and } b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix},$$

this inequalty is referred to as the classical Riesz rearrangement inequality. Theorem 1.27 is one of a family of isoperimetric inequalities: a family which includes the Brascamp-Lieb inequality, found in [50], which in turn has the Loomis-Whitney inequality as a special case, found in [49].

This was extended to inequalities with integrands of the form $F(f_1, \ldots, f_n)$ where F is supermodular; in particular, when F has non-negative mixed second derivatives $\partial_i \partial_j F$ for all $i \neq j$, by Burchard and Hajeiej [14]. A *supermodular* function $F : \mathbb{R}^N \to \mathbb{R}$ is one that satisfies the following inequality:

$$F(y + \alpha e_i + \beta e_j) + F(y) \geq F(y + \alpha e_i) + F(y + \beta e_j), \text{ for all } y \in \mathbb{R}^N,\ i \neq j,\ \alpha,\ \beta > 0$$

where e_i denotes the i-th standard basis vector in $\mathbb{R}^N$.

Ionescu transplanted a modified version of this inequality to non-compact connected semi-simple Lie groups. For the inquisitive reader, these terms are defined in [41]. For clarity, a *Lie group* is a group that is also a differentiable manifold, where the group operations of multiplication and inversion are compatible with the smooth structure, so multiplication and inversion are smooth maps. Every Lie group may be associated with a corresponding Lie algebra, whose underlying vector space is the tangent space of the corresponding Lie group at the identity element. This Lie algebra completely encapluates the local structure of the Lie group. A Lie group is called *semi-simple* if its Lie algebra is semi-simple, that is, it is a direct sum of *simple Lie algebras*, which are non-abelian Lie algebras whose only ideals are the trivial algebra $\{0\}$ and itself. For example, the Lie algebra of the N-dimensional real vector space $\mathbb{R}^N$ is $\mathbb{R}^N$ equipped with the Lie bracket:

$$[x, y] = 0 \text{ for } x, y \in \mathbb{R}^N.$$

Recall that the *centre* of a group is the subgroup of all elements that commute with every other element of the group. A connected semi-simple Lie group has *real rank one* if the abelian subalgebra part of the Iwasawa decomposition of the corresponding Lie algebra has dimension one, as seen in [47]. [43, Theorem 6 (pg. 755)] states

Theorem 1.28. *Let $(\mathbb{G}, \cdot)$ be a non-compact connected semi-simple Lie group with a finite centre. If $\mathbb{G}$ has real rank one then*

$$\int_{\mathbb{G}} \int_{\mathbb{G}} u(g)v(g' \cdot g^{-1})w(g')\ dg\ dg' \leq C \int_{\mathbb{G}} \int_{\mathbb{G}} u^*(g)v^*(g' \cdot g^{-1})w^*(g')\ dg\ dg'$$

for any measurable functions $u, v, w : \mathbb{G} \to \mathbb{R}_+$, for some $C > 0$.

Another avenue of generalisation is to change the codomain of the rearranged functions from the nonnegative reals or $\mathbb{R}$ to $\mathbb{R}^N$, where $N \in \mathbb{N},\ N > 1$. Vector-valued rearrangements have been discussed by Kalton [46], Brenier [10] and R.J. Douglas [25]. All three have used the definition that two functions are rearrangements if the measures of pre-images of the Borel sets of $\mathbb{R}^N$ are equal, which is the characterisation given in Proposition 1.11(ii). Hajaiej also considered the necessity of the assumptions used to prove various rearrangement inequalities, particularly the Hardy-Littlewood and Riesz rearrangement inequalities, in [37]. This paper flows in a similar vein to my work in the next chapter, where I consider the necessity of assumptions to prove properties of the set of rearrangements and its weak closure.

An alternative treatment of rearrangements is given in [51], through the rearrangement of sets. Suppose $A \subset \mathbb{R}^N$, for some $N \in \mathbb{N}$, is a Borel set of finite Lebesgue measure, then the *symmetric rearrangement of the set A* is the open ball $A^\star \subset \mathbb{R}^N$

centred at the origin such that $\mu(A^\star) = \mu(A)$. The symmetric-decreasing rearrangement of a characteristic function is defined as $\mathbb{1}_A^\star = \mathbb{1}_{A^\star}$. For any $f : \mathbb{R}^N \to \mathbb{R}$ that is Borel measurable and vanishing at infinity, the *symmetric-decreasing rearrangement of* f is defined:

$$f^\star : \mathbb{R}^N \to \mathbb{R},\ f^\star(x) = \int_0^\infty \mathbb{1}^\star_{\{y\in\mathbb{R}^N : |f(y)|>t\}}(x)\ \mathrm{d}t.$$

The symmetric-decreasing rearrangement defined above is non-negative, meaning that $f^\star$ is a rearrangement of f, in the sense of Definition 1.1, if and only if f is non-negative. Chapter 3 of [51] focusses on rearrangement inequalities, including:

Theorem 1.29 (Riesz's rearrangement inequality)**.** *Let* $f,\ g,\ h : \mathbb{R}^N \to \mathbb{R}$ *be three non-negative functions, for some* $N \in \mathbb{N}$. *Then, with*

$$I(f, g, h) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x)g(x-y)h(y)\ dx\ dy,$$

we have

$$I(f, g, h) = I(f^\star, g^\star, h^\star),$$

agreeing that $I(f, g, h) = \infty$ *implies that* $I(f^\star, g^\star, h^\star) = \infty$.

Proof. This is [51, Theorem 3.7]. □

The second stream is concerned with particular applications of the theory of rearrangements to physical problems. Elcrat and Nicolio seek steady ideal flows in which the vorticity is a rearrangement of a given function in [28], setting up an iteration that will produce the required rearrangement. This paper also features the K-norm, where K is the linear inverse of the negative Laplacian operator with zero Dirichlet boundary conditions,

$$||\xi||_K = \left(\int_\Omega \xi K\xi\ \mathrm{d}\mu\right)^{\frac{1}{2}},$$

which can be used as an alternate norm on an L^p space. There is a plentitude of papers considering rearrangement optimisation problems, where a rearrangement is sought that maximises a given functional or set of functionals, such as [29, 30] . The motivations are often concerned with fluids; the vorticity in a fixed set of fluid can be thought of as a rearrangement of itself at a later time.

G.R. Burton authored papers with large sections on both the theory of rearrangements and its applications. In [15], investigations are made into a rearrangement that may be written $f^\triangle = \varphi \circ g$, where φ is an increasing function. As the notation suggests, this acts as the increasing rearrangment *with respect to* g. To demonstrate my meaning, [15, Lemma 3 (pg. 230)] states:

Lemma 1.30. *Let (X, Σ, μ) be a finite measure space, let $1 \le p \le \infty$, let q be the conjugate exponent of p, let $\omega = \mu(X)$, let $f \in L^p(X)$ and $g \in L^q(X)$. Suppose f has a rearrangement f^* that satisfies $f^* = \varphi \circ g$ almost everywhere, for some increasing function $\varphi : \mathbb{R} \to \mathbb{R}$. Denote $f^\triangle : [0, \omega] \to \mathbb{R}$ to be the decreasing rearrangement of f on $[0, \omega]$. Then*

$$\int_X f^*(y) g^*(y) \ d\mu(y) = \int_0^\omega f^\triangle(x) g^\triangle(x) \ dx.$$

This generalises the concept of an increasing rearrangement, which had only previously made sense on an ordered domain. An application of the theoretical results in the paper is to vortex rings. A number of results on the properties of rearrangements are collated in [16], such as the convexity and weak sequentially compactness of the weak closure of the set of rearrangements, are given in [16, Lemma 2.2], [16, Lemma 2.3] and [16, Lemma 2.4]. It is mainly these results I am seeking to expunge unnecessary assumptions from in Chapter 2. Given the length of the paper, such generalisations would be better placed in a paper or article on their own. I am also generalising many of the results in this paper, such as [16, Lemma 2.5]:

Lemma 1.31. *Let (X, Σ, μ) be a finite measure space, $1 \le p < \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(X)$, and let $\mathscr{F}$ be the set of rearrangements of f_0 on X. For all $g \in L^q(X)$, define*

$$\eta(g) = \sup_{f \in \mathscr{F}} \int_X fg \ d\mu.$$

Let G be the set of $g \in L^q(\Omega)$ such that $\varphi \circ g \in \mathscr{F}$ for some increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, and let $\kappa(g) = \varphi \circ g$ for this φ. Then

(i) $|\eta(g) - \eta(h)| \le ||f_0||_p ||g - h||_q$ *for all g and $h \in L^q(\Omega)$,*

(ii) $\kappa : G \to \mathscr{F}$ *is strongly continuous.*

A generalisation of the Mountain Pass Lemma [16, Theorem 3.1] for rearrangements forms Chapter 3, and the original form is repeated here:

Theorem 1.32. *Let (Ω, μ) be a measure interval, let $1 < p, q < \infty$ where $p^{-1} + q^{-1} = 1$, let $\Psi : L^p(\Omega) \to \mathbb{R}$ be a continuously differentiable convex functional, let $f_0 \in L^p(\Omega)$, let $\mathscr{F}$ be the set of rearrangements of f_0 on Ω, let $e_0, e_1 \in \mathscr{F}$, and define*

$$\eta(g) = \sup_{f \in \mathscr{F}} \int_X fg \ d\mu \text{ for all } g \in L^q(\Omega).$$

Define

$$
\begin{aligned}
\mathscr{C} &= \{h \in C([0,1], \mathscr{F}) : h(0) = e_0 \text{ and } h(1) = e_1\} \\
c &= \sup_{h \in \mathscr{C}} \inf_{0 \leq t \leq 1} \Psi(h(t)).
\end{aligned}
$$

Suppose

$$\inf \Psi(L^p(\Omega)) < c < \min\{\Psi(e_0), \Psi(e_1)\}.$$

Then there exists a sequence $(v_n)_{n \in \mathbb{N}}$ *in* $\mathscr{F}$ *satisfying*

$$
\begin{aligned}
\lim_{n \to \infty} \Psi(v_n) &= c \\
\lim_{n \to \infty} \eta\left(\Psi'(v_n)\right) - \int_\Omega v_n \Psi'(v_n)\ d\mu &= 0.
\end{aligned}
$$

The application is to a water-flow problem in a dumb-bell domain, where there are two discs connected by a rectangular channel. The thesis will attempt to extend the theoretical knowledge of rearrangements and utilise this new knowledge on physical problems.

Chapter 2

Extensions to the Theory of Rearrangements

2.1 Measure Resolutions and Measure Intervals

Before I begin the investigation in the properties of the set of rearrangements and its weak closure, I shall seek out equivalent properties to a measure space being separable or non-atomic. The aim is to show the set of rearrangements of any real-valued function on a finite and non-atomic measure space is path-connected, among other improvements of existing results.

Definition 2.1 (Atom, Non-atomic Measure Spaces)**.** Let (X, Σ, μ) be a measure space. An *atom* of X is a measurable set $A \in \Sigma$ such that $\mu(A) > 0$ and

$$\text{for every } B \subseteq A \text{ and } B \in \Sigma, \text{ either } \mu(B) = \mu(A) \text{ or } \mu(B) = 0.$$

A measure space is *non-atomic* if it has no atoms.

Definition 2.2 (Separable)**.** Let (X, Σ, μ) be a measure space. X is *separable* if there exists a countable set of measurable sets $\mathscr{D} \subseteq \Sigma$ such that

$$\text{for each } \varepsilon > 0 \text{ and } A \in \Sigma, \text{ there exists } B \in \mathscr{D} \text{ such that } \mu(A \setminus B) + \mu(B \setminus A) < \varepsilon.$$

Remark 2.3 (Nikodym metric space). Alternately, a measure space may be defined via the Nikodym metric space, introduced in [58]. Let (X, Σ, μ) be a finite measure space. Define $A \triangle B = (A \setminus B) \cup (B \setminus A)$, then I introduce a relation $\simeq$ on Σ:

$$A \simeq B \text{ if, and only if } \mu(A \triangle B) = \mu(A \setminus B) + \mu(B \setminus A) = 0.$$

The set identity

$$(A\triangle B)\triangle(B\triangle C) = A\triangle C \text{ holds for all } A, B, C \in \Sigma,$$

implying the relation $\simeq$ is transitive, whilst reflexivity and symmetry are clear from the definition of $\simeq$. Therefore, $\simeq$ is an equivalence relation, and Σ may be decomposed into equivalence classes: denote this collection $\Sigma/\simeq$. For each $A \in \Sigma$, denote the equivalence class of A by $[A]$. On $\Sigma/\simeq$, denote the Nikodym metric by

$$\rho_\mu([A],[B]) = \mu(A\triangle B) \text{ for all } [A],[B] \in \Sigma/\simeq .$$

The metric ρ_μ is properly defined and satisfies the Triangle Inequality. ρ_μ is non-negative because μ is non-negative, and symmetry of the relation implies symmetry of the metric. Also,

$$\rho_\mu([A],[B]) = \mu(A\triangle B) = 0 \Longleftrightarrow [A] = [B].$$

Hence, ρ_μ is indeed on a metric on $\Sigma/\simeq$. It is said that $(\Sigma/\simeq, \rho_\mu)$ is the *Nikodym metric space* associated with (X, Σ, μ) and X is said to be *separable* if $(\Sigma/\simeq, \rho_\mu)$ is separable, that is, it has a dense countable subset.

The notion of a measure interval is consistently used, either explicitly or implicitly, throughout studies into rearrangements and their properties.

Definition 2.4 (Measure Interval)**.** The measure space (X, Σ, μ) is called a *measure interval* if there exists a measure-preserving bijection $\sigma : X \to [0, \mu(X)]$, where $[0, \mu(X)]$ is equipped with the σ-algebra of Lebesgue-measurable subsets of $[0, \mu(X)]$ and the Lebesgue measure.

Theorem 2.5 (Halmos' Isomorphism Theorem)**.** *Suppose (X, Σ, μ) is a measure space, where $0 < \mu(X) < \infty$. (X, Σ, μ) is a measure interval if, and only if, (X, Σ, μ) is a finite, non-atomic and separable measure space.*

Proof. Finite, non-atomic and separable measure spaces with infinite point sets are measure intervals, by [39, Section 41 - Theorem C]. Conversely, suppose (X, Σ, μ) is a measure interval, so there is a measure-preserving bijection $\sigma : X \to [0, \mu(X)]$. Properties of $[0, \mu(X)]$ are preserved by σ, so X is a finite, non-atomic and separable measure space. $\square$

Remark 2.6. If $\mu(X) = 0$, then (X, Σ, μ) is a measure interval, and it is trivially non-atomic. For this reason, there is a restriction to consider only non-trivial measure spaces where $\mu(X) > 0$.

Example 2.7. The prototypical example of a measure interval would be a non-trivial interval $[0, \omega]$, for $\omega > 0$, equipped with the Lebesgue measure. Finite balls in $\mathbb{R}^N$, for some $N \in \mathbb{N}$, are also an example of a measure interval.

Alternate characterisations of a finite measure space being non-atomic or separable are sought. I shall define the *measure resolutions* of a given measurable set with finite measure, and show that it is equivalent to that measure being *convex*, as defined in [38]. Convex measures are equivalent to non-atomic measures.

Definition 2.8 (Measure Resolution)**.** Let (X, Σ, μ) be a measure space. A mapping $\varphi : [0, 1] \to \Sigma$ is called a *μ-resolution* of $A \in \Sigma$, where $\mu(A) < \infty$, if the following properties are satisfied:

(i) $\varphi(t) \subseteq A$ for all $t \in [0, 1]$,

(ii) $0 \le s \le t \le 1$ implies that $\varphi(s) \subseteq \varphi(t)$,

(iii) $\mu(\varphi(t)) = t\mu(A)$ for all $0 \le t \le 1$.

Remark 2.9. The existence of one measure resolution of a measurable set with finite measure begets another. Let $\varphi : [0, 1] \to \Sigma$ be a μ-resolution of $A \in \Sigma$, where $\mu(A) < \infty$. Set

$$\hat{\varphi} : [0, 1] \to \Sigma, \ \hat{\varphi}(t) = A \setminus \varphi(1 - t),$$

then $\hat{\varphi}(t) \subseteq A$ for all $t \in [0, 1]$. Also, if $0 \le s \le t \le 1$, then $\varphi(1 - t) \subseteq \varphi(1 - s)$. It follows that

$$\hat{\varphi}(s) = A \setminus \varphi(1 - s) \subseteq A \setminus \varphi(1 - t) = \hat{\varphi}(t).$$

Lastly, for each $t \in [0, 1]$,

$$\mu\left(\hat{\varphi}(t)\right) = \mu(A \setminus \varphi(1 - t)) = \mu(A) - (1 - t)\mu(A) = t\mu(A).$$

By satisfying the relevant conditions, $\hat{\phi}$ is a μ-resolution of A.

Example 2.10. The clearest example of a measure resolution would be the following resolution of the Lebesgue interval $[0, 1]$, whose measurable subsets are denoted Σ:

$$\varphi : [0, 1] \to \Sigma, \ \varphi(t) = [0, t].$$

There are an uncountably infinite number of measure resolutions of the unit measure interval $[0,1]$. For each $s \in [0,1]$, let

$$\varphi_s : [0,1] \to \Sigma,\ \varphi_s(t) = \left[\frac{1-t}{2}s,\ t + \frac{1-t}{2}s\right],$$

$$\hat{\varphi}_s : [0,1] \to \Sigma,\ \hat{\varphi}_s(t) = \left[0,\ \frac{st}{2}\right] \cup \left[(1-t) + \frac{st}{2},\ 1\right].$$

By inspection, each φ_s and $\hat{\varphi}_s$ for $s \in [0,1]$ is a measure resolution of $[0,1]$.

Measure resolutions are related to the popular concept of a convex measure.

Definition 2.11 (Convex measures)**.** Let (X, Σ, μ) be a finite measure space, and let $E \in \Sigma$. Let $\mathscr{K}(\mu, E)$ denote the class of all real-valued measurable functions $\phi : E \to [0,1)$ such that

$$\mu(\{x \in E : \phi(x) < \lambda\}) = \lambda\mu(E) \text{ for all } \lambda \in [0,1].$$

A measure μ is called *convex* if: for every measurable set $E \in \Sigma$, the class $\mathscr{K}(\mu, E)$ is non-empty.

I discuss the connection between convex measures and measure resolutions of measurable sets in the following result.

Lemma 2.12. *Let (X, Σ, μ) be a finite non-atomic measure space, and fix a measurable set $E \in \Sigma$. Each $\phi \in \mathscr{K}(\mu, E)$ induces a μ-resolution of E. Conversely, each μ-resolution of E induces a measurable function $\phi \in \mathscr{K}(\mu, E)$.*

Proof. Let $\phi : E \to [0,1)$ have the property that $\mu(\{x \in E : \phi(x) < \lambda\}) = \lambda\mu(E)$ for all $\lambda \in [0,1]$. For each $t \in [0,1]$, set

$$\varphi : [0,1] \to \Sigma,\ \varphi(t) = \{x \in E : \phi(x) < t\}.$$

It is clear that $\varphi(t) \subseteq E$. Then $s \leq t$ implies that $\varphi(s) \subseteq \varphi(t)$ and $\mu(\varphi(s)) = s\mu(E)$ for all $s \in [0,1]$. Hence φ is a μ-resolution of E.
Conversely, suppose $\varphi : [0,1] \to \Sigma$ is a μ-resolution of E. Define

$$\phi : E \to [0,1),\ \phi(x) = \inf\{t \in [0,1) : x \in \varphi(t)\} \mathbb{1}_{\varphi([0,1))}(x).$$

Note that, for $s \in [0,1]$,

$$\mu(E \setminus \varphi(s)) = \mu(E) - \mu(\varphi(s)) = (1-s)\mu(E).$$

Then for almost all $x \in E$ and $\lambda \in (0,1]$, we have $\phi(x) < \lambda$ if, and only if, there exists $t \in [0,1)$ such that $t < \lambda$ and $x \in \varphi(t)$, which means that $\phi(x) < \lambda$ if, and only if, $x \in \varphi(\lambda)$. By applying monotone convergence of the measure,

$$\mu(\{x \in E : \phi(x) < \lambda\}) = \mu\left(\bigcup_{0 \le t < \lambda} \varphi(t)\right) = \mu(\varphi(\lambda)) = \lambda\mu(E).$$

Thus, ϕ satifies the conditions of a measurable function in $\mathscr{K}(\mu, E)$. $\square$

Measure resolutions and measure-preserving maps also mutually induce each other.

Lemma 2.13. *Let (X, Σ, μ) be a finite and non-atomic measure space. Each μ-resolution of X induces a measure-preserving map $\sigma : X \to [0, \mu(X)]$ and vice versa. These inducements are inverse operators.*

Proof. Suppose $\varphi : [0,1] \to \Sigma$ is a μ-resolution of X. Set

$$\sigma : X \to [0, \mu(X)],\ \sigma(x) = \inf\{t \in [0,1] : x \in \varphi(t)\}\, \mu(X).$$

Then for each $\alpha \in [0, \mu(X)]$,

$$\sigma^{-1}[0, \alpha] = \varphi\left(\frac{\alpha}{\mu(X)}\right).$$

Hence,

$$\begin{aligned} \mu\left(\sigma^{-1}[0,\alpha]\right) &= \mu\left(\varphi\left(\frac{\alpha}{\mu(X)}\right)\right) \\ &= \frac{\alpha}{\mu(X)}\mu(X) \\ &= \alpha \\ &= \mu_L^{-1}[0,\alpha]. \end{aligned}$$

Therefore, the induced σ is a measure-preserving map from X to $[0, \mu(X)]$. Conversely, suppose $\sigma : X \to [0, \mu(X)]$ is a measure-preserving map. Set

$$\varphi : [0,1] \to \Sigma,\ \varphi(t) = \sigma^{-1}[0, t\mu(X)].$$

Then, for all $t \in [0,1]$, $\varphi(t) = \sigma^{-1}[0, t\mu(X)] \subseteq X$. If $0 \le s \le t \le 1$, then

$$\varphi(s) = \sigma^{-1}[0, s\mu(X)] \subseteq \sigma^{-1}[0, t\mu(X)] = \varphi(t).$$

For $t \in [0,1]$,

$$\begin{aligned}\mu(\varphi(t)) &= \mu\left(\sigma^{-1}[0, t\mu(X)]\right)\\ &= \mu_L[0, t\mu(X)]\\ &= t\mu(X).\end{aligned}$$

Therefore, φ is a μ-resolution of X.
Next, suppose φ is a μ-resolution of X, and denote by $\sigma : X \to [0, \mu(X)]$ the measure-preserving map as induced above. Induce a μ-resolution called $\overline{\varphi}$, by

$$\overline{\varphi} : [0,1] \to \Sigma,\ \overline{\varphi}(t) = \sigma^{-1}[0, t\mu(X)].$$

For each $t \in [0,1]$,

$$\overline{\varphi}(t) = \sigma^{-1}[0, t\mu(X)] = \varphi\left(\frac{t\mu(X)}{\mu(X)}\right) = \varphi(t).$$

Now, suppose $\sigma : X \to [0, \mu(X)]$ is a measure-preserving map. Denote by φ the μ-resolution induced by σ. Then set

$$\overline{\sigma} : X \to [0, \mu(X)],\ \overline{\sigma}(x) = \inf\{t \in [0,1] : x \in \varphi(t)\}\mu(X).$$

Then, for each $x \in X$,

$$\begin{aligned}\overline{\sigma}(x) &= \inf\{t \in [0,1] : x \in \varphi(t)\}\mu(X)\\ &= \inf\left\{t \in [0,1] : x \in \sigma^{-1}[0, t\mu(X)\right\}\mu(X)\\ &= \sigma(x).\end{aligned}$$

Hence, these inducements are inverse operators. $\square$

Next, I show that a measurable set with a measure resolution has the property that all of its measurable subsets also admit measure resolutions. It proves that having a measure resolution is a hereditary property of measurable sets of finite measure.

Proposition 2.14. *Suppose (X, Σ, μ) is a finite measure space, where X admits a μ-resolution, and $Y \in \Sigma$ and $\mu(Y) > 0$. Then (Y, Σ_Y, μ), where*

$$\Sigma_Y = \{A \cap Y : A \in \Sigma\},$$

also admits a μ-resolution.

Proof. Suppose $\varphi : [0,1] \to \Sigma$ is a μ-resolution of X. Now, set

$$\psi : [0,1] \to \Sigma_Y, \; \psi(t) = \varphi(t) \cap Y$$

and define $f : [0,1] \to [0, \mu(Y)], \; f(t) = \mu(\psi(t))$. I seek to show that f is a continuous and increasing function. Let $s, t \in [0,1]$, then $s \leq t$ implies that $\psi(s) \subseteq \psi(t)$, so by monotonicity of the measure, it follows $f(s) \leq f(t)$. Also, for $s, t \in [0,1]$,

$$\psi(s) \triangle \psi(t) = (\varphi(s) \triangle \varphi(t)) \cap Y,$$

and so

$$\begin{aligned} |f(s) - f(t)| &\leq \mu\left(\psi(s) \triangle \psi(t)\right) && \text{by prior calculation} \\ &\leq \mu(\varphi(s) \triangle \varphi(t)) && \text{by monotonicity} \\ &= |s - t| \mu(X). \end{aligned}$$

It follows that f is continuous, increasing and surjective. Now, define:

$$\chi : [0,1] \to \Sigma_Y, \; \chi(t) = \bigcup_{f(s) \leq t\mu(Y)} \psi(s).$$

Then χ is a μ-resolution of Y, as required. $\square$

Remark 2.15. This proof is adapted from [45], where the author called μ-resolution by the name of Σ-*segments*.

Both of these concepts, convex measures and measure resolutions, are equivalent to μ being a non-atomic measure on X on finite measure spaces.

Theorem 2.16. *Let (X, Σ, μ) be a finite measure space. Then the following statements are equivalent:*

(i) (X, Σ, μ) is non-atomic;

(ii) μ is a convex measure on X;

(iii) There is a μ-resolution of X;

(iv) There is a measure-preserving map $\sigma : X \to [0, \mu(X)]$;

(v) There is a measure-preserving map $\sigma : X \to [a, b)$, where $\mu(X) = b - a$;

(vi) Every right-continuous decreasing function on $[0, \mu(X)]$ is the decreasing rearrangement of a measurable function on (X, Σ, μ).

Proof. The strategy for this proof is to show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$.

$\underline{(i) \Rightarrow (ii)}$ μ is convex, by Halmos in [38].

$\underline{(ii) \Rightarrow (iii)}$ By the convexity of μ, then there is a measurable function $\phi : X \to [0, 1)$ with the property that for each $\lambda \in [0, 1]$,

$$\mu(\{x \in X : \phi(x) < \lambda\}) = \lambda\mu(X).$$

This induces a μ-resolution of X by

$$\varphi : [0, 1] \to \Sigma,\ \ \varphi(\lambda) = \{x \in X : \phi(x) < \lambda\},$$

as in Lemma 2.12.

$\underline{(iii) \Rightarrow (iv)}$ Suppose $\varphi : [0, 1] \to \Sigma$ is a μ-resolution of X. Define

$$\sigma : X \to [0, \mu(X)],\ \ \sigma(x) = \inf \{t \in [0, 1] : x \in \varphi(t)\}\mu(X).$$

Then σ is a measure-preserving map.

$\underline{(iv) \Rightarrow (v)}$ Define $\tau : [0, \mu(X)] \to [a, b)$, where $\mu(X) = b - a$, by

$$\tau(x) = \begin{cases} a + t & \text{for } 0 \leq t < \mu(X), \\ a & \text{for } t = \mu(X) \end{cases}$$

Then τ is measure-preserving, and the composition of measure-preserving maps is measure-preserving, so there exists a measure-preserving map $\tau : X \to [a, b)$.

$\underline{(v) \Rightarrow (vi)}$ Let $\sigma : X \to [0, \mu(X)]$ be a measure-preserving map. Suppose $g : [0, \mu(X)] \to \mathbb{R}$ is right-continuous and decreasing, then $f = g \circ \sigma$ is a rearrangement of g, by Proposition 1.20 (vi). It follows that $g = f^{\triangle}$, as right-continuous decreasing rearrangements on $[0, \mu(X)]$ by Lemma 1.14.

$\underline{(vi) \Rightarrow (i)}$ Suppose $f : X \to \mathbb{R}$ is measurable and $f^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ satisfies $f^{\triangle}(t) = \mu(X) - t$ for all $t \in [0, \mu(X)]$. Then f is non-constant on every measurable set with positive measure, so it must mean that (X, Σ, μ) has no atoms. □

Measure resolutions can be utilised to give a slick proof that the product of a finite and non-atomic measure space with an arbitrary finite measure space is also non-atomic.

Proposition 2.17. *Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be two finite measure spaces. If X is non-atomic, then the product measure space $(X \times Y, \Sigma_{X\times Y}, \mu_{X\times Y})$ is non-atomic.*

Proof. By Theorem 2.16, there is a μ-resolution of X, say $\varphi : [0,1] \to \Sigma_X$. Set

$$\psi : [0,1] \to \Sigma_{X\times Y},\ \psi(s) = \varphi(s) \times Y.$$

Then for all $s \in [0,1]$, it follows that $\psi(s) \subseteq X \times Y$. Also, for $0 \leq s \leq t \leq 1$,

$$\psi(s) = \varphi(s) \times Y \subseteq \varphi(t) \times Y = \psi(t).$$

Lastly, for every $\lambda \in [0,1]$,

$$\mu_{X\times Y}(\varphi(\lambda)) = \mu_{X\times Y}(\varphi(\lambda) \times Y) = \mu(\varphi(\lambda))\nu(Y) = \lambda\mu_{X\times Y}(X \times Y).$$

This demonstrates that ψ is a $\mu_{X\times Y}$-resolution of $X \times Y$. Hence, by applying Theorem 2.16 again, $X \times Y$ is non-atomic. $\square$

There is also a useful characterisation of X being separable in terms of the separability of the related $L^p(X)$ spaces.

Lemma 2.18. *Let (X, Σ, μ) be a finite measure space. Then X is separable, if and only if, $L^p(X)$ is separable for $1 \leq p < \infty$.*

Proof. Suppose X is separable, and denote the dense countable algebra of X by $\mathscr{D}$. Then set

$$\mathbb{D} = \left\{ \sum_{i=1}^{n} \alpha_i \mathbb{1}_{A_i} : \alpha_i \in \mathbb{Q},\ A_i \in \mathscr{D},\ n \in \mathbb{N} \right\}.$$

Given $\varepsilon > 0$ and $B \in \Sigma$, we may choose $A \in \mathscr{D}$ such that $\mu(B \triangle A) = \mu(B \setminus A) + \mu(A \setminus B) < \varepsilon$. Then

$$\int_X |\mathbb{1}_B - \mathbb{1}_A|^p \, \mathrm{d}\mu = \mu(B \triangle A) < \varepsilon.$$

Therefore, any simple function may be approximated arbitrarily closely by elements of $\mathbb{D}$, so $\mathbb{D}$ is dense in $L^p(X)$. The converse is equivalent to the denseness of characteristic functions. $\square$

2.2 Properties of the Set of Rearrangements as a Metric Subspace

In the case where the measure space (X, Σ, μ) contains only atoms, the properties of a set of rearrangements over X are often trivial. If a set of rearrangements $\mathscr{F}$ contains only a finite number of functions, then it is also compact. However, it can be shown that a set of rearrangements is not, in general, compact. There are basic properties of this set that can be easily stated.

Definition 2.19 (Hausdorff, Perfectly Normal, Second-Countable, Lindelöf)**.** Let $(Y, \mathscr{T})$ be a topological space. Y is *Hausdorff* if: given two distinct points $x, y \in Y$, where $x \neq y$, there exists open neighbourhoods $U, V \in \mathscr{T}$ with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. Y is *normal* if: given two disjoint closed subsets $A, B \subseteq Y$ such that $A \cap B = \emptyset$, there exists two disjoint open subsets $U, V \in \mathscr{T}$ such that $A \subseteq U,\ B \subseteq V$ and $U \cap V = \emptyset$.
Y is *perfectly normal* if Y is normal and every closed subset is the countable intersection of open subsets.
A topological space Y is *second-countable* if it has a countable base, so there is a countable subset of the open sets $\mathscr{B} \subseteq \mathscr{T}$ such that every open set $U \in \mathscr{T}$ may be expressed as a union of sets from $\mathscr{B}$.
Y is a *Lindelöf* space if every open cover has a countable subcover.

Proposition 2.20. *Let (X, Σ, μ) be a finite measure space, let $1 \leq p < \infty$, and suppose $h \in L^p(X)$. Denote by $\mathscr{F}$ the set of rearrangements of h on X. Then $\mathscr{F}$, treated as a subset of $L^p(X)$, is:*

(*i*) *a metric space;*

(*ii*) *Hausdorff and perfectly normal;*

(*iii*) *closed;*

(*iv*) *a set with empty interior;*

(*v*) *equal to its own boundary;*

(*vi*) *nowhere dense in $L^p(X)$;*

(*vii*) *complete;*

(*viii*) *separable (and so second-countable and Lindelöf);*

(*ix*) *bounded;*

(*x*) *Polish metric space.*

Proof. (*i*) $\mathscr{F}$ may be equipped with the inherited metric from $L^p(X)$:

$$\rho_p(f, g) = \|f - g\|_p \,.$$

(ii) This is immediate from (i).

(iii) Set $g_n = g = h$ in Proposition 1.20 (viii), then for a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ with limit f, then f is a rearrangement of h. Hence, $f \in \mathscr{F}$. Therefore, $\mathscr{F}$ is closed with respect to ρ_p.

(iv) Each open ball $\mathbb{B}(f,\delta)$ in $L^p(X)$ must have non-trivial intersection with $L^p(X)\setminus\mathscr{F}$. Thus, $\mathscr{F}$ has no interior points.

(v) This is immediate from (iii) and (iv):

$$\partial\mathscr{F} = \overline{\mathscr{F}} \setminus \mathscr{F}^\circ = \mathscr{F}.$$

(vi) This also follows from (iii) and (iv):

$$\left(\overline{\mathscr{F}}\right)^\circ = \mathscr{F}^\circ = \emptyset.$$

(vii) This set is a closed subset of a complete metric space $L^p(X)$.

($viii$) As $L^p(X)$ is separable, the subset $\mathscr{F}$ must be separable. The other properties are equivalent as $\mathscr{F}$ is a metric space.

(ix) $\mathscr{F}$ must be bounded, since for all $f, g \in \mathscr{F}$:

$$\rho_p(f,g) = \|f-g\|_p \leq 2\|h\|_p.$$

(x) This follows immediately from (i) and (vii).

$\square$

Many positive properties of the set of rearrangements are derived from its metrizability. The set of rearrangements is perfectly normal and Hausdorff, so it satisfies strong separation axioms.

Compactness is a commonly considered property of topological spaces. However, the set of rearrangements is not, in general, totally bounded. Let $X = [0,1]$ be the unit Lebesgue measure, and suppose we are considering the set of rearrangments of the identity map $\mathrm{id} : [0,1] \to [0,1]$, denoted $\mathscr{F}$. Set the sequence

$$f_n : [0,1] \to [0,1], \; f_n(x) = nx \bmod 1.$$

Each member of this sequence is a rearrangement of the identity map, that is, $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}$. The sequence $(f_n)_{n\in\mathbb{N}}$ weakly converges to a constant function, so it cannot converge strongly to a function in $\mathscr{F}$. Hence, $\mathscr{F}$ is not, in general, totally bounded. Compact metric spaces are complete and totally bounded, so $\mathscr{F}$ cannot be, in general, compact or sequentially compact. This counterexample also demonstrates that the set of rearrangements is not generally locally compact either.

Definition 2.21 (Paracompact, Refinement, Locally Finite)**.** Let $(Y, \mathscr{T})$ be a topological space. Let $\mathscr{A}$ be the index of a cover, so there exists a family of subsets of Y, $\mathbf{U} = \{U_\alpha : \alpha \in \mathscr{A}\}$, such that $Y \subseteq \bigcup_{\alpha\in\mathscr{A}} U_\alpha$. A *refinement* of a given cover is a family of subsets of Y, $\mathbf{V} = \{V_\beta : \beta \in \mathscr{B}\}$ for some index $\mathscr{B}$, such that, $Y \subseteq \bigcup_{\beta\in\mathscr{B}} V_\beta$ and for all $\beta \in \mathscr{B}$, there exists an $\alpha \in \mathscr{A}$ such that $V_\beta \subseteq U_\alpha$. This refinement is called *open* if it is comprised of only open subsets.
A cover $\mathbf{U} = \{U_\alpha : \alpha \in \mathscr{T}\}$ is *locally finite* if for any $y \in Y$, there exists an open neighbourhood $y \in V_y$ with $V_y \in \mathscr{T}$ such that the set

$$\{\alpha \in \mathscr{A} : U_\alpha \cap V_y \neq \emptyset\}$$

is finite.
Y is paracompact if every open cover has an open refinement that is locally finite.

Corollary 2.22. *Let (X, Σ, μ) be a finite measure space, let $1 \le p < \infty$, and suppose $h \in L^p(X)$. Denote by $\mathscr{F}$ the set of rearrangements of h on X. Then $\mathscr{F}$, equipped with inherited metric from $L^p(X)$, is paracompact.*

Proof. This is a classical result: all metric spaces are paracompact, as in [63]. □

It can be said that the set of rearrangements does not satisfy many compactness conditions. This failure is one of the motivating factors in considering the weak closure of the set of rearrangments.
However, the set of rearrangements on a finite and non-atomic measure space meets many conditions of connectedness. If (X, Σ, μ) is comprised of two atoms of equal measure, then if $h : X \to \mathbb{R}$ is non-constant, then the set of rearrangements of h on X has only two functions, and so is trivially disconnected. Finally, it is shown that if the underlying measure space is finite and non-atomic, then all p-integrable functions, where $1 \le p < \infty$, have contractible sets of rearrangements on that measure space, where the set of rearrangements is seen as a subset of the L^p-space.

Theorem 2.23. *Suppose (X, Σ, μ) is a finite and non-atomic measure space, let $1 \le p < \infty$ and let $h_0 : X \to \mathbb{R}$. Denote by $\mathscr{F}$ the set of rearrangements of h_0 on X. Then $\mathscr{F} \subset L^p(X)$ is contractible.*

Proof. Assume, without loss of generality, that the range of h is a subset of $(0,1)$. Otherwise, define the map

$$\tau : \mathbb{R} \to (0,1),\ \ \tau(s) = \frac{1}{2} + \frac{1}{\pi}\arctan(s),$$

as in [16, Lemma 2.11], and relabel f as $\tau \circ f$ for each $f \in \mathscr{F}$. This map τ is continuous and preserves two functions being rearrangements. At the end of the argument, compose τ^{-1} with $\tau \circ f$ to yield the correct homotopy in $\mathscr{F}$, which uses that τ^{-1} is also continuous. Let $h \in \mathscr{F}$ be arbitrary. Since X is finite and non-atomic, Theorem 2.16 (iv) means there exists a measure-preserving map $\sigma : X \to [0, \mu(X)]$ such that $h = h_0^{\triangle} \circ \sigma$, where $h_0^{\triangle} : [0, \mu(X)] \to \mathbb{R}$ is the decreasing rearrangement of h_0.

This measure-preserving map σ may induce a measurable function $\phi : X \to [0,1)$, with the property that for each $\lambda \in [0,1]$,

$$\mu\left(\{x \in X : \phi(x) < \lambda\}\right) = \lambda\mu(X).$$

Set $\phi : X \to [0,1),\ \phi(x) = \mu(X)^{-1}\sigma(x)$. Let $\lambda \in [0,1]$, then

$$\begin{aligned}
\mu\left(\{x \in X : \phi(x) < \lambda\}\right) &= \mu\left(\{x \in X : \sigma(x) < \lambda\mu(X)\}\right) \\
&= \mu\left(\sigma^{-1}[0, \lambda\mu(X)\right) \\
&= \mu_L\left([0, \lambda\mu(X))\right) \\
&= \lambda\mu(X).
\end{aligned}$$

Hence, this function ϕ has the desired property. By Theorem 2.16 (iii), this ϕ induces a μ-resolution called φ. By unions and complements of negligible sets, ensure that $\varphi(0) = \emptyset$ and $\varphi(1) = X$, as $\mu(\varphi(0)) = 0$ and $\mu(\varphi(1)) = \mu(X)$.
For $\lambda \in [0,1]$ and measurable $f : X \to \mathbb{R}$, set

$$f_\lambda(x) = \begin{cases} f(x) & \text{for } x \in \varphi(\lambda), \\ 0 & \text{for } x \notin \varphi(\lambda). \end{cases}$$

Note that $f_0 = 0$ and $f_1 = f$. By assumption, the essential range of all functions is a subset of $(0,1)$, so

$$f_\lambda^{\triangle}(x) = \begin{cases} f|_{\varphi(\lambda)}^{\triangle}(x) & \text{for } x \in [0, \lambda\mu(X)], \\ 0 & \text{for } x \in (\lambda\mu(X), \mu(X)] \end{cases}$$

Hence,

$$f_\lambda = f_\lambda^{\triangle} \circ \sigma \text{ for all } \lambda \in [0,1].$$

Construct the mapping

$$H : \mathscr{F} \times [0,1] \to \mathscr{F}, \ H(f,t) = f_t^{\triangle} \circ \sigma + f - f_t.$$

By construction, $H(f,t) \in \mathscr{F}$ for all $(f,t) \in \mathscr{F} \times [0,1]$, so H is well-defined. Also,

$$\begin{aligned} H(f,0) &= f = \mathrm{id}_{\mathscr{F}}(f), \\ H(f,1) &= f_1^{\triangle} \circ \sigma + f - f_1 = f_1^{\triangle} \circ \sigma = h_0^{\triangle} \circ \sigma = h. \end{aligned}$$

Now, it is only required to show that H is continuous. Let $\varepsilon > 0$ and $(f,t) \in \mathscr{F} \times [0,1]$ be arbitrary.

Suppose $(g,s) \in \mathscr{F} \times [0,1]$. By the construction of H, and the non-expansive properties of the decreasing rearrangement, it follows that:

$$\begin{aligned} \|H(f,t) - H(g,s)\|_p &\leq \left\| f_t^{\triangle} \circ \sigma + f - f_t - g_s^{\triangle} \circ \sigma - g + g_s \right\|_p \\ &\leq \left\| f_t^{\triangle} \circ \sigma - g_s^{\triangle} \circ \sigma \right\|_p + \|f_t - g_s\|_p + \|f - g\|_p \\ &\leq 2 \|f_t - g_s\|_p + \|f - g\|_p . \end{aligned}$$

Now, a bound of $\|f_t - g_s\|_p$ in terms of ε is sought. Note $s \leq t$ if, and only if, $\varphi(s) \subseteq \varphi(t)$. Assume, without loss of generality, that $s \leq t$. Then $x \in \varphi(s)$ implies that $f_t(x) = f(x)$ and $g_s(x) = g(x)$,and $x \notin \varphi(t)$ implies that $f_t(x) = g_s(x) = 0$.

In the case that $x \in \varphi(t) \setminus \varphi(s)$, $f_t(x) = f(x)$ and $g_s(x) = 0$, and $\int_{\varphi(t)\setminus\varphi(s)} |f|^p \, \mathrm{d}\mu$ is bounded above by $\int_0^{t-s} \left(|h_0|^{\triangle}\right)^p \, \mathrm{d}\mu_L$. Generalising this working yields:

$$\begin{aligned} \int_X |f_t - g_s|^p \, \mathrm{d}\mu &= \int_{\varphi(s)\cup\varphi(t)} |f_t - g_s|^p \, \mathrm{d}\mu \\ &\leq \int_{\varphi(s)\cap\varphi(t)} |f - g|^p \, \mathrm{d}\mu + \int_0^{|s-t|} \left(|h_0|^{\triangle}\right)^p \, \mathrm{d}\mu_L. \end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, there exists $\delta > 0$ such that

$$|t - s| < \delta \text{ implies that } \int_0^{|s-t|} \left(|h_0|^{\triangle}\right)^p \, \mathrm{d}\mu_L < (2^p - 1) \left(\frac{\varepsilon}{5}\right)^p.$$

Then for $|t-s|<\delta$ and $\|f-g\|_p < \frac{\varepsilon}{5}$, it follows

$$\begin{aligned}
\|H(f,t)-H(g,s)\|_p &\leq 2\|f_t - g_s\|_p + \|f-g\|_p \\
&< 2\left(\frac{\varepsilon^p}{5^p} + (2^p-1)\frac{\varepsilon^p}{5^p}\right)^{\frac{1}{p}} + \frac{\varepsilon}{5} \\
&= 2\left(\frac{2^p\varepsilon^p}{5^p}\right)^{\frac{1}{p}} + \frac{\varepsilon}{5} \\
&= \varepsilon.
\end{aligned}$$

Hence, H is continuous, and $\mathscr{F}$ is a contractible set. $\square$

Example 2.24. I shall consider the case of the identity map on the unit interval, $\mathrm{id}_{[0,1]} : [0,1] \to [0,1],\ \mathrm{id}_{[0,1]}(x) = x$. Let $\mathscr{F}$ denote the rearrangements of $\mathrm{id}_{[0,1]}$ on $[0,1]$. A continuous path between $\mathrm{id}_{[0,1]}$ and $f : [0,1] \to [0,1],\ f(x) = 1-x$. For each $s \in [0,1]$,

$$f_s : [0,1] \to [0,1],\ f_s(x) = \begin{cases} x & \text{if } x \in [0,s] \\ 1+s-x & \text{if } x \in (s,1] \end{cases}.$$

By setting

$$\gamma : [0,1] \to \mathscr{F},\ \gamma(s) = f_s.$$

This defines a continuous path in $L^\infty([0,1])$, and hence every $L^p([0,1])$ for $1 \leq p < \infty$, where $f_0 = f$ and $f_1 = \mathrm{id}_{[0,1]}$.

It can also be shown that the set of rearrangements, where the underlying measure space is finite and non-atomic, is locally contracible.

Proposition 2.25. *Suppose (X, Σ, μ) is a finite and non-atomic measure space, let $1 \leq p < \infty$ and let $h_0 \in L^p(X)$. Denote by $\mathscr{F}$ the set of rearrangements of h_0 on X. Then $\mathscr{F}$ is locally contractible.*

Proof. Assume, without loss of generality, that the range of h_0 is a subset of $(0,1)$. Here, I will use the notation given in Theorem 2.23.
Let $g \in \mathscr{F}$, and suppose $\sigma : X \to [0, \mu(X)]$ is the measure-preserving bijection such that $g = h_0^\triangle \circ \sigma$. Since $g = h_0^\triangle \circ \sigma$, it follows that $g_t = g_t^\triangle \circ \sigma$ for each $t \in [0,1]$. Let $\varepsilon > 0$ be arbitrary, and set $\delta = \frac{\varepsilon}{2}$. I seek to show that

$$H : \mathbb{B}(g,\delta) \times [0,1] \to \mathbb{B}(g,\varepsilon),\ H(f,t) = f_t^\triangle \circ \sigma + f - f_t$$

is a valid homotopy. The continuity of H follows from a similar calculation given in Theorem 2.23. It holds that $H(f,0) = f = \mathrm{id}_{\mathscr{F}}(f)$ and $H(f,1) = h_0^\triangle \circ \sigma = g$. The remaining calculation is to show that $H(f,t) \in \mathbb{B}(g,\varepsilon)$ for all $(f,t) \in \mathbb{B}(g,\delta) \times [0,1]$.

Suppose $(f,t) \in \mathbb{B}(g,\delta) \times [0,1]$. It follows, by the Triangle Inequality of norms, the measure-preserving properties of σ and the non-expansive properties of decreasing rearrangements, that

$$\begin{aligned}
\|H(f,t) - g\|_p &= \left\|(f_t^{\triangle} \circ \sigma - g_t^{\triangle} \circ \sigma)\mathbb{1}_{\varphi(t)} + (f-g)\mathbb{1}_{X\setminus\varphi(t)}\right\|_p \\
&\leq \left\|f_t^{\triangle} \circ \sigma - g_t^{\triangle} \circ \sigma\right\|_p + \|f-g\|_p \\
&= \left\|f_t^{\triangle} - g_t^{\triangle}\right\|_p + \|f-g\|_p \\
&\leq \|f_t - g_t\|_p + \|f-g\|_p \\
&\leq 2\|f-g\|_p < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.
\end{aligned}$$

Hence, H is a valid homotopy, and $\mathscr{F}$ is locally contractible. $\square$

A corollary of Theorem 2.23 is that the set of rearrangements is path-connected, when the underlying measure space is finite and non-atomic.

Theorem 2.26. *Suppose (X, Σ, μ) is a finite and non-atomic measure space, let $1 \leq p < \infty$ and let $h_0 : X \to \mathbb{R}$. Denote by $\mathscr{F}$ the set of rearrangements of h_0 on X. Then $\mathscr{F}$ is path-connected.*

Proof. Let $g \in \mathscr{F}$ be arbitrary. By Theorem 2.23, there exists a continuous function $H : \mathscr{F} \times [0,1] \to \mathscr{F}$ such that $H(f,0) = f$ and $H(f,1) = g$ for all $f \in \mathscr{F}$. Now, let $f \in \mathscr{F}$ be arbitrary, then the function

$$\gamma : [0,1] \to \mathscr{F}, \ \gamma(t) = H(f,t),$$

is a continuous path from $\gamma(0) = H(f,0) = f$ and $\gamma(1) = H(f,1) = g$. It follows that $\mathscr{F}$ is path-connected. $\square$

Remark 2.27. Note that the proof of Theorem 2.26 may be generalised, as all contractible sets are path-connected.

The converse statement - if every set of rearrangements on X is path-connected, then X is non-atomic - is false. Consider the following measure space: $X = \{0\} \cup [2,3]$, $\Sigma = \{A : A \in \mathscr{B}([2,3])\} \cup \{A \cup \{0\} : A \in \mathscr{B}([2,3])\}$ and the measure μ is defined by $\mu(A) = \mu_L(A)$, the Lebesgue measure for $A \in \mathscr{B}([2,3])$ and $\mu(\{0\}) = 2$. Here, X is a finite measure space with a single atom $\{0\}$. Since $\mu(\{0\}) = 2 > 1 = \mu([2,3])$, any measurable function $f : X \to \mathbb{R}$ must remain constant on the atom $\{0\}$. Let $f : X \to \mathbb{R}$ be measurable, and denote $\mathscr{F}$ to be the set of rearrangements of f on X. Suppose

$g \in \mathscr{F}$. Assume, seeking contradiction, that $g|_{\{0\}} \neq f|_{\{0\}}$, then

$$3 \geq \mu\left(\left\{x \in X : g(x) = g|_{\{0\}}(x)\right\}\right) + \mu\left(\left\{x \in X : g(x) = f|_{\{0\}}(x)\right\}\right) \geq 2 + 2 = 4,$$

which is absurd. It follows that $g|_{\{0\}} = f|_{\{0\}}$. By Proposition 1.20 $(xiii)$, $\mathscr{F}$ may be expressed as:

$$\mathscr{F} = \left\{g \in \mathscr{F} : g|_{\{0\}} = f|_{\{0\}},\ g|_{[2,3]} \text{ is a rearrangement of } h|_{[2,3]}\right\}.$$

Since the interval $[2, 3]$ is a finite and non-atomic measure space, Theorem 2.26 implies $\mathscr{F}$ is path-connected. The measure space X has an atom, but the set of rearrangements of all meaurable functions $f : X \to \mathbb{R}$ are path-connected, so the converse is false.

There are further corollaries of these results.

Corollary 2.28. *Let (X, Σ, μ) be a finite and non-atomic measure space, let $1 \leq p < \infty$ and let $h_0 : X \to \mathbb{R}$. Denote by $\mathscr{F}$ the set of rearrangements of h on X_0. Then $\mathscr{F} \subset L^p(X)$ is connected, locally connected, locally path-connected, simply connected, locally simply connected, uniformly locally connected and well-chained.*

Proof. $\mathscr{F}$ is contractible, so it is both connected, simply connected and well-chained. This latter property is considered in [70, Definition (p. 10)].
$\mathscr{F}$ is locally contractible, and hence locally path-connected, locally simply connected. Uniform locally connectedness can be shown in the following manner: Let $\varepsilon > 0$ be arbitrary, and set $\delta = 2\epsilon > 0$. If $f, g \in \mathscr{F}$ satisfy $\|f - g\|_p < \varepsilon$, then $f, g \in \mathbb{B}(f, \delta) \cap \mathscr{F}$, which is connected, open in $\mathscr{F}$ and has diameter $2\varepsilon = \delta$. □

Corollary 2.29. *Let (X, Σ, μ) be a finite and nonatomic measure space, let $1 \leq p < \infty$, let $f_0 \in L^p(X)$, let $\mathscr{F}$ be the set of rearrangements of f_0 on X. Then the fundamental group $\pi_1(\mathscr{F})$ is trivial.*

Proof. According to Theorem 2.23, every path, and so every loop, in $\mathscr{F}$ may be continuously deformed into any other. This is precisely the statement that the fundamental group is trivial. □

There are other properties of topological spaces that cannot be easily categorised.

Definition 2.30 (Homogeneous)**.** Let Y be a topological space. Y is said to be *homogeneous* if for every $x, y \in Y$, there exists a homeomorphism $\psi : Y \to Y$ such that $\psi(x) = y$.

Lemma 2.31. *Let (X, Σ, μ) be a finite and non-atomic measure space and let $1 \leq p < \infty$. Suppose $\sigma : X \to X$ is a measure-preserving transformation, and define*

$$T : L^p(X) \to L^p(X),\ Tf = f \circ \sigma.$$

Then:

(i) T is linear;

(ii) T is norm-preserving under the p-norm, that is,

$$\|Tf\|_p = \|f\|_p \text{ for all } f \in L^p(X);$$

(iii) T is injective;

(iv) T is a bijection if, and only if, σ is a measure-preserving bijection.

(v) if T is a bijection and $\mathscr{F}$ is a set of rearrangements, then $T(\mathscr{F}) = \mathscr{F}$.

Proof. (i) Let $\alpha, \beta \in \mathbb{R}$ and $f, g \in L^p(X)$, then

$$(\alpha f + \beta g) \circ \sigma = \alpha(f \circ \sigma) + \beta(g \circ \sigma);$$

so $T(\alpha f + \beta g) = \alpha T f + \beta T g$.

(ii) Let $f \in L^p(X)$, then

$$\|Tf\|_p = \left(\int_X (|f|^p \circ \sigma) \, \mathrm{d}\mu \right)^{\frac{1}{p}} = \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} = \|f\|_p \,.$$

(iii) As T is linear, T injects if, and only if, $\ker T = \{0\}$. This holds by (ii), as

$$f \in \ker T \Longleftrightarrow Tf = 0 \Longleftrightarrow \|f\|_p = \|Tf\|_p = 0 \Longleftrightarrow f = 0.$$

Hence, T is injective.

(iv) Suppose $\sigma : X \to X$ is a measure-preserving bijection, then for each $f \in L^p(X)$,

$$T(f \circ \sigma^{-1}) = (f \circ \sigma^{-1}) \circ \sigma = f.$$

Hence, T surjects, and is thus bijective by (iii).

(v) Let $\mathscr{F}$ denote the set of rearrangements of f_0 on X, where $f_0 \in L^p(X)$. Then for each $f \in \mathscr{F}$, $Tf = f \circ \sigma \in \mathscr{F}$ and $g = f \circ \sigma^{-1} \in \mathscr{F}$ satisfies $Tg = f$ up to negligible sets. It follows that $T(\mathscr{F}) = \mathscr{F}$.

If $\sigma : X \to X$ fails to be a measure-preserving bijection, then it must fail to be injective. For each $f \in L^p(X)$, then $Tf = f \circ \sigma$ is not injective, up to negligible sets. There must exist injective functions in $L^p(X)$ on a non-atomic measure space, so T cannot be bijective.

$\square$

There are doubly stochastic functions which take one function in $\mathscr{F}$ to another.

Lemma 2.32. *Suppose (X, Σ, μ) is a finite and non-atomic measure, and let $1 \leq p < \infty$. For each $f, g \in L^p(X)$, there exists a doubly stochastic map $T : L^p(X) \to L^p(X)$ such that $Tf = g$.*

Proof. This is [64, Lemma 3]. $\square$

Remark 2.33. In general, the set of rearrangements on a finite and non-atomic measure space fails to be homogeneous. I will use the example given in [64]. Consider $X = [0, 1]$ and suppose $h : [0, 1] \to [0, 1],\ h(s) = 1 - s$. Denote the set of rearrangements of h on $[0, 1]$ by $\mathscr{F}$.

h is already decreasing, and so it must be that $h = h^{\triangle}$. Every function where $f \in \mathscr{F}$ may be characterised as $f = h^{\triangle} \circ \sigma = \sigma$, where $\sigma : [0, 1] \to [0, 1]$ is a measure-preserving map. The function $T : L^1([0, 1]) \to L^1([0, 1])$ induced by

$$\tau : [0, 1] \to [0, 1],\ \tau(s) = 2s \bmod 1.$$

For the reasons described in Lemma 2.31, any map linking an arbitrary f to $f \circ \tau$ would not be a homeomorphism.

In [11], it is shown that the set $\{\sigma : [0, 1] \to [0, 1] : \sigma \text{ is a measure-preserving bijection}\}$ is norm-dense in $\{\sigma : [0, 1] \to [0, 1] : \sigma \text{ is a measure-preserving map}\}$. This former set is plainly not equal to $\mathscr{F}$, as there some measure-preserving maps which are not bijective.

2.3 Further Properties of Rearrangements and the Set of Rearrangements

For each desirable property of rearrangements and the set of rearrangements, I will investigate what suppositions are required to imply each property. Most of the results had used that the underlying measure space is a measure interval, and so, finite, non-atomic and separable. I assume throughout that the measure space is finite, so I wish to see if the measure space also needs to be non-atomic, separable, neither or both. The first issue is to verify that standard inequalities for rearrangements hold in finite and non-atomic measure spaces, rather than just holding for measure intervals.

The famous Hardy-Littlewood inequality holds for all pairs of measurable functions on finite measure spaces.

Theorem 2.34 (Hardy-Littlewood Inequality)**.** *Let (X, Σ, μ) be a finite measure space, with $\omega = \mu(X)$. Let $f, g : X \to \mathbb{R}$ be two measurable functions such that $|f|^{\triangle}$, $|g|^{\triangle} \in L^2([0, \mu(X)])$. Then $fg \in L^1(X)$ and*

$$\int_0^{\omega} f^{\triangle}(\omega - t) g^{\triangle}(t)\ dt = \int_0^{\omega} f^{\triangle}(t) g^{\triangle}(\omega - t)\ dt \leq \int_X fg\ d\mu \leq \int_0^{\omega} f^{\triangle}(t) g^{\triangle}(t)\ dt.$$

Proof. The proof of the inequality for characteristic functions is quickly verified, and is a standard calculation. Another short calculation and linearity of the integral shows it is valid for simple functions. Non-negative measurable functions may be approximated by increasing sequences of simple functions, so the Dominated Convergence Theorem shows the inequality holds for non-negative functions. Let $f, g : X \to \mathbb{R}$ satisfy $|f|^{\triangle}, |g|^{\triangle} \in L^2([0, \omega])$. Since the inequality has been proven for non-negative functions, the case for functions with both signs must be proved.
It follows

$$0 \leq \int_0^{\omega} |f|^{\triangle}(\omega - t) g^{\triangle}(t)\ \mathrm{d}t \leq \int_X |f(x)||g(x)|\ \mathrm{d}\mu(x) \leq \int_0^{\omega} |f|^{\triangle}(t)|g|^{\triangle}(t)\ \mathrm{d}t < \infty.$$

It is immediate that $fg \in L^1(X)$. Now, denote for a measurable function $h : X \to \mathbb{R}$, $h_1 = h_+$ and $h_2 = h_-$,

$$h^{\triangle}(t) = h_1^{\triangle}(t) + (-h_2)^{\triangle}(t) = h_1^{\triangle}(t) - h_2^{\triangle}(\omega - t) \text{ for } 0 \leq t \leq \omega.$$

The integrals may be delineated as:

$$\begin{aligned}
\int_0^{\omega} f^{\triangle}(t) g^{\triangle}(\omega - t)\mathrm{d}t &= \int_0^{\omega} f_1^{\triangle}(t) g_1^{\triangle}(\omega - t)\ \mathrm{d}t - \int_0^{\omega} f_2^{\triangle}(t) g_1^{\triangle}(t)\ \mathrm{d}t \\
&\quad - \int_0^{\omega} f_1^{\triangle}(t) g_2^{\triangle}(t)\ \mathrm{d}t + \int_0^{\omega} f_2^{\triangle}(\omega - t) g_2^{\triangle}(t)\ \mathrm{d}t, \\
\int_0^{\omega} f^{\triangle}(t) g^{\triangle}(t)\mathrm{d}t &= \int_0^{\omega} f_1^{\triangle}(t) g_1^{\triangle}(t)\ \mathrm{d}t - \int_0^{\omega} f_2^{\triangle}(\omega - t) g_1^{\triangle}(t)\ \mathrm{d}t \\
&\quad - \int_0^{\omega} f_1^{\triangle}(t) g_2^{\triangle}(\omega - t)\ \mathrm{d}t + \int_0^{\omega} f_2^{\triangle}(t) g_2^{\triangle}(t)\ \mathrm{d}t, \\
\int_X fg\ \mathrm{d}\mu &= \int_X f_1 g_1\ \mathrm{d}\mu - \int_X f_2 g_1\ \mathrm{d}\mu - \int_X f_1 g_2\ \mathrm{d}\mu + \int_X f_2 g_2\ \mathrm{d}\mu.
\end{aligned}$$

By the case for nonnegative functions, we have that

$$\begin{aligned}
\int_0^\omega f_1^\triangle(t)g_1^\triangle(\omega-t)\ \mathrm{d}t &\le \int_X f_1g_1\ \mathrm{d}\mu \le \int_0^\omega f_1^\triangle(t)g_1^\triangle(t)\ \mathrm{d}t,\\
-\int_0^\omega f_2^\triangle(t)g_1^\triangle(t)\ \mathrm{d}t &\le -\int_X f_2g_1\ \mathrm{d}\mu \le -\int_0^\omega f_2^\triangle(\omega-t)g_1^\triangle(t)\ \mathrm{d}t,\\
-\int_0^\omega f_1^\triangle(t)g_2^\triangle(t)\ \mathrm{d}t &\le -\int_X f_1g_2\ \mathrm{d}\mu \le -\int_0^\omega f_1^\triangle(t)g_2^\triangle(\omega-t)\ \mathrm{d}t,\\
\int_0^\omega f_2^\triangle(\omega-t)g_2^\triangle(t)\ \mathrm{d}t &\le \int_X f_2g_2\ \mathrm{d}\mu \le \int_0^\omega f_2^\triangle(t)g_2^\triangle(t)\ \mathrm{d}t.
\end{aligned}$$

By the case for non-negative functions, and the linearity of integrals, the result follows:

$$\int_0^\omega f^\triangle(t)g^\triangle(\omega-t)\ \mathrm{d}t \le \int_X f(x)g(x)\ \mathrm{d}\mu(x) \le \int_0^\omega f^\triangle(t)g^\triangle(t)\ \mathrm{d}t.$$

□

Theorem 2.35. *Let (X,Σ,μ) be a finite and non-atomic measure space. Let $J:\mathbb{R}\to[0,\infty)$ be a non-negative and convex function such that $J(0)=0$. Let $f,g:X\to[0,\infty)$ be non-negative measurable functions, and fix $\sigma:X\to[0,\mu(X)]$, a measure-preserving map. Set $f^*=f^\triangle\circ\sigma,\ g^*=g^\triangle\circ\sigma$. Then $f^*,\ g^*:X\to[0,\infty)$ and*

$$\int_X J(f^*-g^*)\ d\mu \le \int_X J(f-g)\ d\mu.$$

Proof. Note that the non-atomic supposition is being used by the existence of a measure-preserving map $\sigma:X\to[0,\mu(X)]$, by Theorem 2.16 (*iii*). I extend the proof given in [51, Theorem 3.5]. First, write $J=J_++J_-$, where $J_+(t)=0$ for $t\le 0$ and $J_+(t)=J(t)$ for $t\ge 0$, and similarly for J_-. Both of these functions are convex, so the result may be proved for these two functions separately. Now, I write J instead of J_+ or J_-. Since J is convex, it has a right derivative $J'(t)$ for all $t\in\mathbb{R}$ and $J(t)=\int_0^t J'(s)\ \mathrm{d}s$. The convexity of J implies that $J'(t)$ is an increasing function of $t\in[0,\infty)$. Hence, for each $x\in X$,

$$J(f(x)-g(x)) = \int_{g(x)}^{f(x)} J'(f(x)-s)\ \mathrm{d}s = \int_0^\infty J'(f(x)-s)\mathbb{1}_{\{y\in X:g(y)\le s\}}(x)\ \mathrm{d}s.$$

It follows, by integrating over X and then by Fubini's Theorem, that

$$\begin{aligned}\int_X J(f(x)-g(x))\ \mathrm{d}\mu(x) &= \int_X \int_0^\infty J'(f(x)-s)\mathbb{1}_{\{y\in X: g(y)\le s\}}(x)\ \mathrm{d}s\ \mathrm{d}\mu(x)\\ &= \int_0^\infty \int_X J'(f(x)-s)\mathbb{1}_{\{y\in X: g(y)\le s\}}(x)\ \mathrm{d}\mu(x)\ \mathrm{d}s.\end{aligned}$$

For each $s \in (0,\infty)$,

$$\int_X J'(f(x)-s)\mathbb{1}_{\{y\in X: g(y)\le s\}}(x)\ \mathrm{d}\mu(x) \ge \int_X J'(f^*(x)-s)\mathbb{1}_{\{y\in X: g^*(y)\le s\}}(x)\ \mathrm{d}\mu(x).$$

This inequality may be proved by imitating the proof for Theorem 2.34, as stated on [51, Remark on p. 75].

Hence, it follows that

$$\int_X J(f^*-g^*)\ \mathrm{d}\mu \le \int_X J(f-g)\ \mathrm{d}\mu.$$

□

A corollary of this result is that the map which sends p-integrable functions over finite and non-atomic measure spaces, for $1 \le p < \infty$, to their decreasing rearrangement is non-expansive.

Corollary 2.36. *Let (X,Σ,μ) be a finite and non-atomic measure space, and let $f,\ g : X \to \mathbb{R}$ be measurable functions. Suppose for $1 \le p < \infty$ and $f,\ g \in L^p(X)$, and denote $f^\triangle,\ g^\triangle : [0,\mu(X)] \to \mathbb{R}$ to be the decreasing rearrangements of f and g respectively, then*

$$\left\| f^\triangle - g^\triangle \right\|_p \le \|f-g\|_p.$$

Proof. The case of non-negative functions $f, g \ge 0$, since $J : \mathbb{R} \to \mathbb{R},\ J(t) = |t|^p$ is a non-negative convex function such that $J(0) = 0$ and an application of Theorem 2.35. The case where f and g are bounded below is an immediate consequence of the non-negative case. As in [16, Lemma 2.7], set for each $\alpha < 0$,

$$\begin{aligned}f_\alpha &: X \to \mathbb{R},\ f_\alpha(x) = \max\{\alpha, f(x)\},\\ g_\alpha &: X \to \mathbb{R},\ g_\alpha(x) = \max\{\alpha, g(x)\}.\end{aligned}$$

Hence, for each $\alpha < 0$, it is found that

$$\left\| f_\alpha^\triangle - g_\alpha^\triangle \right\|_p \le \|f_\alpha - g_\alpha\|_p.$$

By the Dominated Convergence Theorem, it also holds for the pointwise limits as $\alpha \to -\infty$, where the pointwise limits of $f_\alpha^\triangle, g_\alpha^\triangle, f_\alpha, g_\alpha$ are $f^\triangle, g^\triangle, f, g$ respectively. Hence,

$$\left\| f^\triangle - g^\triangle \right\|_p \leq \| f - g \|_p ,$$

as required. $\square$

Next, I improve a result in [64], showing that if the measure space is finite and non-atomic, there exists a sufficient variety of measure-preserving maps from X to $[0, \mu(X)]$ that every measurable function can be expressed as a composition of its decreasing rearrangement with a measure-preserving map.

Theorem 2.37. *Let (X, Σ, μ) be a finite and non-atomic measure space, and let $f : X \to \mathbb{R}$ be a measurable function. Then there is a measure-preserving map $\sigma : X \to [0, \mu(X)]$ such that $f = f^\triangle \circ \sigma$ μ-almost everywhere.*

Proof. This theorem is found in [64], but was only previously considered to be true for intervals. Set for each $t \in \mathbb{R}$,

$$I_t = \left\{ s \in [0, \mu(X)] : f^\triangle(s) = t \right\} \text{ and } X_t = \{ x \in X : f(x) = t \} .$$

By Corollary 1.12, $\mu(X_t) = \mu_L(I_t)$ for each $t \in \mathbb{R}$, where μ_L is the Lebesgue measure on $[0, \mu(X)]$. Since $f^\triangle$ and f are rearragements, f can be replaced by a rearrangement f' such that $f' = f$ μ-almost everywhere and f' and $f^\triangle$ have the same range Y, by Proposition 1.20 (i). As $f^\triangle$ is decreasing and right-continuous, each I_t has either the form $[a, b]$ or $[a, b)$. Hence, for each $t \in Y$, there is a measure-preserving map $\sigma_t : X_t \to I_t$, where each σ_t is surjective. These measure-preserving maps exist because each X_t is a non-atomic measurable set, and applying Theorem 2.16 (iv). Define

$$\sigma : X \to [0, \mu(X)], \ \sigma(x) = \sigma_t(x), \ x \in X_t.$$

Clearly, $f = f^\triangle \circ \sigma$. It remains to show that σ is measure-preserving. Since $\mu(X) < \infty$, then $\mu_L(I_t) > 0$ for at most countably many such $t \in \mathbb{R}$. Let Z be the set of all $t \in \mathbb{R}$ such that $\mu_L(I_t) > 0$, and let $I = \bigcup_{t \in Z} I_t$, which is measurable since the complement of I is, and on which $f^\triangle$ injects. Let $A \subseteq [0, \mu(X)]$ be measurable, then

$$A = (A \cap I) \cup (A \setminus I) = \bigcup_{t \in Z} (A \cap I_t) \cup (A \setminus I),$$

from which it follows

$$\mu(\sigma^{-1}(A)) = \sum_{t \in Z} \mu\left(\sigma^{-1}(A \cap I_t)\right) + \mu\left(\sigma^{-1}(A \setminus I)\right).$$

But $\mu\left(\sigma^{-1}(A \cap I_t)\right) = \mu\left(\sigma_t^{-1}(A \cap I_t)\right) = m(A \cap I_t)$, and $f^{\triangle}$ injects onto $A \setminus I$, so $f = f^{\triangle} \circ \sigma$ implies that $\sigma = (f^{\triangle})^{-1} \circ f$ on $A \setminus I$, so

$$\mu\left(\sigma^{-1}(A \setminus I)\right) = \mu\left(f^{-1}\left(f^{\triangle}(A \setminus I)\right)\right) = \mu_L\left(\left(f^{\triangle}\right)^{-1}\left(f^{\triangle}(A \setminus I)\right)\right) = \mu_L(A \setminus I).$$

Hence, σ is measure-preserving. □

Corollary 2.38. *Suppose (X, Σ, μ) is a finite measure space, and let $f : X \to \mathbb{R}$. Then X is also non-atomic if, and only if, there exists a measure-preserving map $\sigma : X \to [0, \mu(X)]$ such that $f = f^{\triangle} \circ \sigma$ μ-almost everywhere.*

Proof. If X is non-atomic, then Theorem 2.37 implies that there exists such a measure-preserving map $\sigma : X \to [0, \mu(X)]$. If there exists a measure-preserving map $\sigma : X \to [0, \mu(X)]$ such that $f = f^{\triangle} \circ \sigma$ almost everywhere in X, then the mere existence of that map is the condition (iv) in Theorem 2.16, meaning that X is non-atomic. □

Remark 2.39. Along with the conditions detailed in Theorem 2.16, I have given 7 conditions that are equivalent to X being non-atomic, when X is also finite.

A prepatory result is required for showing that finite and non-atomic measure spaces have path-connected sets of rearrangements, as it shows the integral is absolutely continuous.

Proposition 2.40. *Let (X, Σ, μ) be a measure space and the function $f : X \to \mathbb{R}$ be integrable. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that for each measurable subset $E \subseteq X$,*

$$\text{if } \mu(E) < \delta, \text{ then } \int_E |f| \; d\mu < \varepsilon.$$

Furthermore, for each $\varepsilon > 0$, there is a subset X_0 of X that has finite measure and satisfies

$$\int_{X \setminus X_0} |f| \; d\mu < \varepsilon.$$

Proof. This is a classical result, as seen in [62, Section 18.3, Proposition 17].

G.R. Burton has provided a short proof of this proposition: let $Y \in \Sigma$, with $y = \mu(Y)$, then

$$\int_Y |f| \; \mathrm{d}\mu \leq \int_0^y |f|^{\triangle}(t) \; \mathrm{d}t,$$

and by the Monotone Convergence Theorem,

$$0 \leq \lim_{y \to 0} \int_Y |f| \; \mathrm{d}\mu \leq \lim_{y \to 0} \int_0^y |f|^{\triangle}(t) \; \mathrm{d}t = 0.$$

Thus, the integral in question can be made arbitrarily small. □

Given that $\mathscr{F}$ is path-connected for functions over finite and non-atomic measure spaces, this result can be combined with the Hardy-Littlewood inequality to produce the following theorem.

Theorem 2.41. *Suppose (X, Σ, μ) is a finite and non-atomic measure space, with $\omega = \mu(X)$. Denote two functions being rearrangements by the symbol $\sim$. If $f,\ g : X \to \mathbb{R}$ are measurable functions such that $|f|^{\triangle},\ |g|^{\triangle} \in L^1[0, \omega]$, then*

$$\left\{ \int_X fg' \; d\mu : g' \sim g \right\} = \left[\int_0^{\omega} f^{\triangle}(t) g^{\triangle}(\omega - t) \; dt, \int_0^{\omega} f^{\triangle}(t) g^{\triangle}(t) \; dt \right].$$

Proof. Fix $f,\ g : X \to \mathbb{R}$ and denote $\mathscr{F}$ to be the set of rearrangements of g on X. Then the functional

$$H : \mathscr{F} \to \mathbb{R},\ H(v) = \int_X fv \; \mathrm{d}\mu$$

is Lipschitz continuous with respect to the standard metric. By the Hardy-Littlewood Inequality, given by Theorem 2.34, it follows

$$\left\{ \int_X fg' \; \mathrm{d}\mu : g' \sim g \right\} \subseteq \left[\int_0^{\omega} f^{\triangle}(t) g^{\triangle}(\omega - t) \; \mathrm{d}t,\ \int_0^{\omega} f^{\triangle}(t) g^{\triangle}(t) \; \mathrm{d}t \right].$$

Since $\mathscr{F}$ is path-connected by Theorem 2.26, the continuous image of $\mathscr{F}$ under H is an interval in $\mathbb{R}$. It is only required to show the upper and lower bounds are attained. If $\sigma : X \to [0, \omega]$ satisfies $f = f^{\triangle} \circ \sigma$, which must exist by Theorem 2.37, then setting $g' = g^{\triangle} \circ \sigma \sim g$ yields

$$\begin{aligned} \int_X fg' \; \mathrm{d}\mu &= \int_X (f^{\triangle} g^{\triangle}) \circ \sigma \; \mathrm{d}\mu && \text{by construction of } g' \\ &= \int_0^{\omega} f^{\triangle}(t) g^{\triangle}(t) \; \mathrm{d}t && \text{by Proposition 1.20 } (vi) \text{ and } (x). \end{aligned}$$

The other bound has a similar proof, so the result is completed. □

Next, I consider what it means for a finite measure space to be *adequate*.

Definition 2.42 (Adequate)**.** Suppose (X, Σ, μ) is a finite measure space, where $\omega = \mu(X)$. Two measurable functions $u, v : X \to \mathbb{R}$ being rearrangements is denoted $u \sim v$.

The measure space X is called *adequate* if for all non-negative functions $f,\ g : X \to \mathbb{R}$:

$$\max\left\{\int_X fg' \ \mathrm{d}\mu : g' \sim g\right\} = \int_0^\omega f^{\triangle}(t) g^{\triangle}(t) \ \mathrm{d}t.$$

Remark 2.43. When X is a finite measure space which consists solely of atoms of equal measure, it is often described as *discrete*. Suppose X has n atoms of equal measure. Since every measurable function is constant on atoms, each function may be identified with an n-dimensional real vector. Measure-preserving maps from X into X are induced by the permutations of the atoms, which is equivalent to bijections from $\{1, \ldots, n\}$ to itself. A similar study called the *rearrangement of vectors* is precisely what the theory of rearrangements of functions is generalising.

Adequate measure spaces may also be called *resonant*, as in [8].

Example 2.44. Sliding puzzles and rubix cubes are examples of rearrangements of vectors. The nature of these puzzles is that only specific types of rearrangements may be used to transform one configuration into another. In a sliding puzzle, only tiles adjacent to the blank tile may be swapped with that blank tile.

$$\begin{pmatrix} 1 & 3 & 6 \\ 4 & & 8 \\ 5 & 7 & 2 \end{pmatrix} \text{ is rearranged to } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{pmatrix}.$$

The nature of adequate measure spaces is discussed in [21].

Theorem 2.45. *Let (X, Σ, μ) be a finite measure space, with $\omega = \mu(X)$. Then the following conditions are equivalent:*

(i) (X, Σ, μ) is adequate;

(ii) (X, Σ, μ) is non-atomic or consists solely of atoms of equal measure;

(iii) For all $A, B \in \Sigma$,

$$\begin{aligned}\sup\left\{\int_X \mathbb{1}_A \mathbb{1}_E \ d\mu : \mathbb{1}_E \sim \mathbb{1}_B\right\} &= \sup\left\{\mu(A \cap E) : \mu(E) = \mu(B)\right\} \\ &= \int_0^\omega \mathbb{1}_A^{\triangle}(t) \mathbb{1}_B^{\triangle}(t) \ dt.\end{aligned}$$

Proof. I follow [21] here. The strategy for the proof is $(ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii)$.
$\underline{(ii) \Rightarrow (i)}$ Suppose (X, Σ, μ) is non-atomic. By Theorem 2.41, it follows for non-negative

measurable functions $f,\ g : X \to \mathbb{R}$ that

$$\max\left\{\int_X fg'\ \mathrm{d}\mu : g' \sim g\right\} = \int_0^\omega f^\triangle(t)g^\triangle(t)\ \mathrm{d}t.$$

Hence, X is adequate. The proof for when X is discrete is similar.

<u>*(i)* $\Rightarrow$ *(iii)*</u> This implication follows from considering the defining property in terms of characteristic functions.

<u>*(iii)* $\Rightarrow$ *(ii)*</u> Assume, seeking contradiction, that *(ii)* is false. Either X has at least two atoms $A, B \in \Sigma$ with $0 < \mu(B) < \mu(A)$, or X has a solitary atom A and non-atomic part X_0, both of positive measure. Hence, there exists a measurable set $B \subseteq X_0$ such that $0 < \mu(B) < \mu(A)$. In either case, for all $E \in \Sigma$ with $\mathbb{1}_E \sim \mathbb{1}_B$, then $\mu(E) = \mu(B)$, and

$$\mu(A \cap E) \leq \mu(E) = \mu(B) < \mu(A).$$

As A is an atom, so $\mu(A \cap E) = 0$, but

$$\int_0^\omega \mathbb{1}_A^\triangle(t)\mathbb{1}_B^\triangle(t)\ \mathrm{d}t = \min\{\mu(A), \mu(B)\} = \mu(B) > 0.$$

This is a contradiction, so the implication holds. $\square$

Remark 2.46. The failure of condition *(ii)* in Theorem 2.45 can mean that a measure space has atoms with distinct measures, or say, two atoms of equal measure and a non-trivial non-atomic part.

In order to study the weak closure of the set of rearrangments of a given function, I must re-introduce a pre-order relation originally given in [40], and generalised in [21].

Definition 2.47 (Majorised)**.** Let (X, Σ, μ) be a finite measure space, with $\omega = \mu(X)$. Let $f, g : X \to \mathbb{R}$ be two measurable functions, with $|f|^\triangle,\ |g|^\triangle \in L^1(X)$. The function g is said to be *majorised* by f, denoted $g \prec f$, if

$$\int_0^s g^\triangle(t)\ \mathrm{d}t \leq \int_0^s f^\triangle(t)\ \mathrm{d}t \text{ for all } 0 \leq s \leq \omega \text{ and } \int_X g\ \mathrm{d}\mu = \int_X f\ \mathrm{d}\mu.$$

Remark 2.48. Note that, by Proposition 1.20 *(x)*,

$$\int_0^\omega f^\triangle(t)\ \mathrm{d}t = \int_X f\ \mathrm{d}\mu \text{ and } \int_0^\omega g^\triangle(t)\ \mathrm{d}t = \int_X g\ \mathrm{d}\mu.$$

This means that the majorised pre-ordering is a condition on the decreasing rearrangements of each function.

Lemma 2.49. *Let (X, Σ, μ) be a finite measure space, with $\omega = \mu(X)$, and let $1 \leq p < \infty$. Then Orb$(f) = \{g \in L^p(X) : g \prec f\}$ is the closed convex hull of*

$$\mathscr{F} = \{g \in L^p(X) : g \text{ is a rearrangement of } f\}$$

for all $f \in L^p(X)$ if, and only if, (X, Σ, μ) is adequate.

Proof. P.W. Day proved this in [21, Theorem 5.3]. □

I need to characterise the extreme points of the set $\text{Orb}(f) = \{g \in L^p(X) : g \prec f\}$, which Ryff succeeded in doing in [64, 66].

Theorem 2.50. *Let (X, Σ, μ) be a finite measure space, $1 \leq p < \infty$, and suppose $f \in L^p(X)$. Then $g \in L^p(X)$ is a rearrangement of f if, and only if, g is an extreme point of $\{h \in L^p(X) : h \prec f\}$.*

Proof. This is contained in [64, 66]. □

Proposition 2.51. *Let (X, Σ, μ) be a measure interval, and let $1 \leq p < \infty$. For each $f \in L^p(X)$, the set of rearrangements of f on X is weakly dense in $\{g \in L^p(X) : g \prec f\}$.*

Proof. Ryff proved this result in [65, Proposition 1]. □

It is required that X is a measure interval for the set of rearrangements of f on X to be weakly dense in the set $\{g \in L^p(X) : g \prec f\}$, so if the underlying measure space is a measure interval, then the weak closure of this set of rearrangements is convex.

Lemma 2.52. *Suppose (X, Σ, μ) is a measure interval, and let $f \in L^p(X)$ for $1 \leq p < \infty$. Then $\overline{\mathscr{F}^w}$, the weak closure of the set of rearrangements of f on X, is convex.*

Proof. By Proposition 2.51, the set of rearrangements of f on X is weakly dense in $\{g \in L^p(X) : g \prec f\}$, so $\overline{\mathscr{F}^w}$, the weak closure of the set of rearrangements of f on X, is equal to $\{g \in L^p(X) : g \prec f\}$. Since this set is the closed convex hull of $\mathscr{F}$ by Lemma 2.49, $\overline{\mathscr{F}^w}$ is convex. □

Remark 2.53. Set (X, Σ, μ) to be the measure space in Remark 2.27, then this measure space also demonstrates that the converse statement to Lemma 2.52 - if every weak closure of the set of rearrangements on X is convex, then X is a measure interval - is false.

Now, I consider the other properties of the weak closure of the set of rearrangements, but this will require the Riesz Weak Compactness Theorem and the Dunford-Pettis Theorem.

Theorem 2.54 (Riesz Weak Compactness Theorem)**.** *Let (X, Σ, μ) be a σ-finite measure space and $1 < p < \infty$. Then every bounded sequence in $L^p(X)$ has a weakly convergent subsequence; that is, if $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(X)$, then there is a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and a function f in $L^p(X)$ for which*

$$\lim_{i\to\infty} \int_X f_{n_i} g \; d\mu = \int_X fg \; d\mu \text{ for all } g \in L^q(X), \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Since $1 < p < \infty$, we have that $L^p(X)$ is reflexive, thanks to [62, Chapter 14 - Proposition 20]. Every bounded sequence in a reflexive Banach space has a weakly convergent subsequence. The conclusion now follows from the Riesz Representation Theorem for the dual of $L^p(X)$. □

I require the definition of uniformly integrable family of measurable functions, as given in [62, Definition on p. 93].

Definition 2.55 (Uniformly integrable)**.** Let (X, Σ, μ) be a finite measure space. A family $\mathscr{G}$ of measurable functions on X is called *uniformly integrable* (or *equiintegrable*) over X if: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $g \in \mathscr{G}$,

$$\text{if } A \in \Sigma \text{ and } \mu(A) < \delta, \text{ then } \int_A |g| \, \mathrm{d}\mu < \varepsilon.$$

Theorem 2.56 (Dunford-Pettis Theorem)**.** *For a finite measure space (X, Σ, μ) and a bounded sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1(X)$, the following two properties are equivalent:*

(i) $(f_n)_{n\in\mathbb{N}}$ is uniformly integrable over X;

(ii) Every subsequence of $(f_n)_{n\in\mathbb{N}}$ has a further subsequence that converges weakly in $L^1(X)$.

Proof. This was shown in [62]. □

Theorem 2.57. *Let X be a measure space and let $1 < p < \infty$. Then every bounded sequence in $L^p(X)$ has a subsequence that converges weakly in $L^p(X)$ to a function in $L^p(X)$.*

Proof. This result is [62, Theorem 14 (pg. 171)]. □

Lemma 2.58. *Let (X, Σ, μ) be a finite measure space, let $1 \le p \le \infty$, let q be the conjugate exponent of p, let $f_0 \in L^p(X)$ and let $\mathscr{F}$ denote the set of rearrangements of f_0 on X. Let $\overline{\mathscr{F}^w}$ denote the closure of $\mathscr{F}$ in the weak topology on $L^p(X)$. If X is separable and $1 < p < \infty$, then $\overline{\mathscr{F}^w}$ is weakly compact. If X is separable and $p = \infty$,* *then $\overline{\mathscr{F}^w}$ is weak-$*$ sequentially compact. Also, $1 \leq p < \infty$ implies $\overline{\mathscr{F}^w}$ is weakly sequentially compact.*

Proof. I restate the proof of [15, Lemma 6]. For all $f \in \mathscr{F}$, Proposition 1.20 (x) implies $||f||_p = ||f_0||_p$, hence $\overline{\mathscr{F}^w}$ is bounded. When $1 < p < \infty$, then $L^p(X)$ is reflexive, according to [62, Proposition 20 (pg. 284)]. Every closed, bounded, convex subset of a reflexive Banach space is weakly compact. Thus, $\overline{\mathscr{F}^w}$ is weakly compact.
The separability of (X, Σ, μ) ensures that $L^1(X)$ is separable, so if $p = \infty$, then $\overline{\mathscr{F}^w}$ is a bounded weak-$*$ closed set in the dual of separable Banach space, hence $\overline{\mathscr{F}^w}$ is weak-$*$ sequentially compact. Consider finally the case $p = 1$. We have

$$\lim_{M \to \infty} \int_{|f(x)| \geq M} |f| \, \mathrm{d}\mu = 0$$

uniformly over $f \in \mathscr{F}$. The weak sequential compactness of $\overline{\mathscr{F}^w}$ now follows from the Dunford-Pettis criterion for weak compactness in $L^1(X)$. If $1 < p < \infty$, then Theorem 2.57 states every bounded sequence in $L^p(X)$ has a weakly convergent subsequence, meaning that bounded sets in $L^p(X)$ are weakly sequentially compact. Rearrangements have equal norms by Proposition 1.20 (xi), so $\overline{\mathscr{F}^w}$ is bounded in $L^p(X)$, meaning the weak closure of the set of rearrangements is weakly sequentially compact. □

Lastly, I seek to find necessary conditions for $\overline{\mathscr{F}^w}$ with the weak topology to be metrizable.

Theorem 2.59. *Let (X, Σ, μ) be a finite and separable measure space. Suppose $f_0 : X \to \mathbb{R}$ is a measurable function, $1 < p < \infty$ and $\mathscr{F} \subset L^p(X)$ is the set of rearrangements of f_0 on X. Then $\overline{\mathscr{F}^w}$ with the weak topology is metrizable.*

Proof. The result [32, Proposition 3.107] states that if $\mathbb{D}$ is a countable separating set in $(L^p(X))^*$, then the weak topology on $\overline{\mathscr{F}^w}$ is metrizable and coincides with the strong topology on $\overline{\mathscr{F}^w}$, as $\overline{\mathscr{F}^w}$ is weakly compact. Such a separating set exists if $L^p(X)$ is separable, which is equivalent to X being separable by Lemma 2.18. □

I amalgamate the prior results into the main result of this subsection.

Theorem 2.60. *Suppose (X, Σ, μ) is a finite measure space, with $\omega = \mu(X) > 0$. Let $f_0 \in L^p(X)$ be a function on X with $1 \leq p < \infty$, and denote the set of rearrangements of f_0 on X by $\mathscr{F}$. The weak closure of $\mathscr{F}$ is denoted $\overline{\mathscr{F}^w}$. Then*

(i) for $f, g \in L^p(X)$, it follows

$$\int_0^\omega f^\triangle(t) g^\triangle(\omega - t) \ dt \leq \int_X fg \ d\mu \leq \int_0^\omega f^\triangle(t) g^\triangle(t) \ dt;$$

(ii) *if* X *is non-atomic, then for* $f, g \in L^p(X)$*, it follows*

$$\left\| f^{\triangle} - g^{\triangle} \right\|_p \leq \| f - g \|_p ;$$

(iii) X *is non-atomic implies* $\mathscr{F}$ *is path-connected;*

(iv) *if* X *is non-atomic, then for each* $g \in \mathscr{F}$*, there exists a measure-preserving map* $\sigma : X \to [0, \mu(X)]$ *such that* $g = f_0^{\triangle} \circ \sigma$*;*

(v) $\overline{\mathscr{F}^w}$ *is weakly sequentially compact;*

(vi) X *is separable and* $1 < p < \infty$ *implies* $\overline{\mathscr{F}^w}$ *is weakly compact;*

(vii) X *is a measure interval implies* $\overline{\mathscr{F}^w}$ *is convex;*

(viii) *if* X *is separable and* $1 < p < \infty$*, then* $\overline{\mathscr{F}^w}$ *is metrizable.*

Proof. (*i*) was proved in Theorem 2.34. (*ii*) is the content of Corollary 2.36. In (*iii*), we showed $\mathscr{F}$ was path-connected for finite, non-atomic measure spaces in Theorem 2.26. Theorem 2.37 is (*iv*), and Lemma 2.58 forms (*v*) and (*vi*). Now, (*vii*) was proved in Lemma 2.52. Finally, (*viii*) follows from Theorem 2.59. $\square$

Chapter 3

Generalising the Mountain Pass Lemma

3.1 Preparatory Results

The aim of this chapter is to generalise the Mountain Pass Lemma given in [16, Theorem 3.1], taking it from considering only paths of rearrangements to continuous maps from the unit disc to the set of rearrangements. I shall begin by proving some preparatory results.

Lemma 3.1. *Let $(Y, \|\cdot\|_Y)$ be a uniformly convex Banach space, let $(u_n)_{n\in\mathbb{N}} \subset Y$ and let $u \in Y$. Suppose that $(u_n)_{n\in\mathbb{N}}$ converges weakly to u and that $\limsup_{n\to\infty} ||u_n||_Y = ||u||_X$. Then $(u_n)_{n\in\mathbb{N}}$ converges strongly to u.*

Proof. This is a simple consequence of the Hahn-Banach Theorem. □

The proof of the next theorem shall be split into two cases: $1 < p < \infty$ and $p = 1$. The latter shall also require that the underlying finite measure space (X, Σ, μ) is non-atomic. We shall use the following notation: if $f \in L^p(X)$ and $g \in L^q(X)$, where q is the conjugate exponent of p, then

$$\langle f, g\rangle = \int_X fg \ \mathrm{d}\mu.$$

Theorem 3.2. *Let (X, Σ, μ) be a finite measure space, let $1 < p < \infty$, let $p^{-1} + q^{-1} = 1$, let $f_0 \in L^p(X)$, let $g \in L^q(X)$, and let $\mathscr{F}$ be the set of rearrangements of f_0 on X. Suppose there is an increasing function φ such that $f^* = \varphi \circ g \in \mathscr{F}$. Suppose $(f_n)_{n\in\mathbb{N}}$*

is a sequence in $L^p(X)$ *such that*

$$\lim_{n\to\infty} \left\| f_n^{\triangle} - f_0^{\triangle} \right\|_p = 0,$$
$$\lim_{n\to\infty} \langle f_n, g \rangle = \langle f^*, g \rangle .$$

Then

$$\lim_{n\to\infty} \|f_n - f^*\|_p = 0.$$

Proof. Assume, seeking contradiction, that $(f_n)_{n\in\mathbb{N}}$ does not converge to f^* in $L^p(X)$. Hence, for some $\varepsilon_0 > 0$, there exists a subsequence $(f_{n_l})_{l\in\mathbb{N}}$ such that

$$\text{for all } l \in \mathbb{N},\ ||f_{n_l} - f^*||_p \geq \varepsilon_0.$$

For convenience, denote the converging subsequence by $(f_n)_{n\in\mathbb{N}}$. Since $||f_n^{\triangle}||_p = ||f_n||_p$ for all $n \in \mathbb{N} \cup \{0\}$, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(X)$. $\overline{\mathscr{F}^w}$, the weak closure of $\mathscr{F}$, is weakly sequentially compact, thanks to Theorem 2.60 (v). Hence, there exists a weakly convergent subsequence $(f_{n_i})_{i\in\mathbb{N}}$, with weak limit $h \in L^p(X)$.
Since $\lim_{i\to\infty} ||f_{n_i}^{\triangle} - f_0^{\triangle}||_p = 0$, $\mathscr{F}$ is the set of rearrangements of f_0 on X, and h is the weak limit of $(f_{n_i})_{i\in\mathbb{N}}$, then I conclude that $h \in \overline{\mathscr{F}^w}$, the weak closure of $\mathscr{F}$. Then

$$\lim_{i\to\infty} \langle f_{n_i}, g \rangle = \langle h, g \rangle = \langle f^*, g \rangle .$$

It follows from [16, Lemma 2.4 (c)] that f^* is the unique maximiser of the linear functional $\langle \cdot, g \rangle$ relative to $\overline{\mathscr{F}^w}$, so I deduce $h = f^*$. $L^p(X)$ is uniformly convex, which is shown in [18]. The conclusion is, by Lemma 3.1, that $\lim_{i\to\infty} \|f_{n_i} - f^*\|_p = 0$. This is a contradiction. □

Remark 3.3. This proof may not be extended to $p = 1$, because we have used that $L^p(X)$ is uniformly convex, which does not hold when $p = 1$. Generally, the above result uses that a sequence with the property that every subsequence has its own subsequence that converges to a particular point must converge to that point.

The next case, $p = 1$, requires that the underlying measure space is finite and non-atomic.

Theorem 3.4. *Let* (X, Σ, μ) *be a finite and non-atomic measure space, where* $\omega = \mu(X)$. *Suppose* $f_0 \in L^1(X)$, $g \in L^\infty(X)$ *and let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $L^1(X)$. *Suppose* $\mathscr{F}$ *is the set of rearrangements of* f_0 *on* X. *Set* $f^* = \varphi \circ g \in \mathscr{F}$, *where* φ *is an increasing* *function, then if*

$$\lim_{n\to\infty} \left\| f_n^{\triangle} - f_0^{\triangle} \right\|_1 = 0,$$
$$\textit{and} \lim_{n\to\infty} \langle f_n, g \rangle = \langle f^*, g \rangle .$$

Then

$$\lim_{n\to\infty} \|f_n - f^*\|_1 = 0.$$

Proof. Let $(\sigma_n)_{n\in\mathbb{N}}$ be the sequence of measure-preserving maps from X to the measure interval $[0, \omega]$ such that $f_n = f_n^{\triangle} \circ \sigma_n$. These maps exists due to Theorem 2.37. Set

$$f_n^* = f_0^{\triangle} \circ \sigma_n \in \mathscr{F}, \text{ for all } n \in \mathbb{N}.$$

Then, $(f_n^*)_{n\in\mathbb{N}}$ is a sequence that satisfies, due to the properties of measure-preserving maps,

$$\lim_{n\to\infty} ||f_n^* - f_n||_1 = \lim_{n\to\infty} ||f_0^{\triangle} \circ \sigma_n - f_n^{\triangle} \circ \sigma_n||_1 = \lim_{n\to\infty} ||f_0^{\triangle} - f_n^{\triangle}||_1 = 0.$$

It also follows,

$$\lim_{n\to\infty} |\langle f_n^* - f_n, g \rangle| \le \lim_{n\to\infty} ||f_n^* - f_n||_1 ||g||_\infty = 0.$$

Hence, the sequence $(f_n^*)_{n\in\mathbb{N}}$ converges to $f^* \in \mathscr{F}$ in $L^1(\Omega)$ by [15, Theorem 2]. By the Triangle Inequality,

$$\lim_{n\to\infty} ||f_n - f^*||_1 = 0.$$

□

Remark 3.5. This proof may also be used in the case that $1 < p < \infty$, changing 1 to p and ∞ to q, where $p^{-1} + q^{-1} = 1$, assuming that the underlying measure space is non-atomic.

A similar form of the following Lemma was used in the proof of [16, Theorem 3.1]. The underlying measure space was only assumed to be finite in that result, but for this version, it is required that it is also non-atomic when $p = 1$. Now, I show that a metric can be imposed under on the point set of sets of rearrangements.

Lemma 3.6. *Suppose (X, Σ, μ) is a measure space and let $1 \le p \le \infty$. Denote by $\overline{f} \subset L^p(X)$ the equivalence class of $f \in L^p(X)$ under the equivalence relation that $g \sim f$ if, and only if, g is a rearrangment of f. Let $Y = \left\{\overline{f} : f \in L^p(X)\right\}$ and set the metric of Y to be:*

$$\rho_Y(\overline{f}, \overline{h}) = \left\| f^{\triangle} - h^{\triangle} \right\|_p .$$

Then (Y, ρ_Y) is a metric space.

Proof. First, the metric is non-negative, as norms are non-negative. The Triangle Inequality of this metric is immediate from the Triangle Inequality in the p-norm. Clearly, it is also symmetric. It is positive definite because, for all $\overline{f}, \overline{h} \in Y$,

$$\rho_Y(\overline{f}, \overline{h}) = 0 \iff f^{\triangle} = h^{\triangle} \iff \overline{f} = \overline{h}.$$

$\square$

Lemma 3.7. *Let (X, Σ, μ) be a finite and non-atomic measure space, with $\omega = \mu(X)$. Let $1 \leq p < \infty$, let q be the conjugate exponent of p. Let $f_0 \in L^p(X)$, denote the set of rearrangements of f_0 on X by $\overline{f_0}$. Let $Y = \left\{\overline{f} : f \in L^p(X)\right\}$, with the metric $\rho_Y(\overline{f}, \overline{h}) = \left\|f^{\triangle} - h^{\triangle}\right\|_p$.*
For all $\overline{f} \in Y$ and $g \in L^q(X)$, define $\eta(g, \overline{f})$ to the supremum of $\langle \cdot, g\rangle$ relative to $\overline{f}$. Let $G \subseteq L^q(X) \times Y$ be the set of all pairs $(g, \overline{f})$ such that $g \in L^q(X)$, $\varphi \circ g \in \overline{f}$ for some increasing function φ, and let $\kappa(g, \overline{f}) = \varphi \circ g$. Then

(i) $\left|\eta(g_1, \overline{f_1}) - \eta(g_2, \overline{f_2})\right| \leq ||f_1||_p||g_1 - g_2||_q + ||g_2||_q \rho_Y(\overline{f_1}, \overline{f_2})$ *for all* $\overline{f_1}, \overline{f_2} \in Y$ *and* $g_1, g_2 \in L^q(X)$,

(ii) $\kappa : G \to L^p(X)$ *is well-defined and strongly continuous.*

Proof. (*i*) Note for all $\overline{f} \in Y$ and $g \in L^q(X)$, we have that

$$\eta(g, \overline{f}) = \int_0^{\omega} f^{\triangle} g^{\triangle} \, \mathrm{d}\mu_L.$$

It follows that:

$$\begin{aligned}
\left|\eta(g_1, \overline{f_1}) - \eta(g_2, \overline{f_2})\right| &= \left|\int_0^{\omega} f_1^{\triangle} g_1^{\triangle} \, \mathrm{d}\mu_L - \int_0^{\omega} f_2^{\triangle} g_2^{\triangle} \, \mathrm{d}\mu_L\right| \\
&= \left|\int_0^{\omega} f_1^{\triangle} \left(g_1^{\triangle} - g_2^{\triangle}\right) \, \mathrm{d}\mu_L + \int_0^{\omega} \left(f_1^{\triangle} - f_2^{\triangle}\right) g_2^{\triangle} \, \mathrm{d}\mu_L\right| \\
&\leq ||f_1^{\triangle}||_p ||g_1^{\triangle} - g_2^{\triangle}||_q + ||f_1^{\triangle} - f_2^{\triangle}||_p ||g_2^{\triangle}||_q \\
&\leq ||f_1||_p |g_1 - g_2||_q + ||g_2||_q \rho_Y(\overline{f_1}, \overline{f_2}).
\end{aligned}$$

These final two inequalities hold due to the Triangle Inequality, and the definition of ρ_Y.

(*ii*) If $(g, \overline{f}) \in G$, then $\kappa(g, \overline{f})$ is well-defined, since $\kappa(g, \overline{f})$ maximises $\langle \cdot, g\rangle$ relative to $\overline{f}$, and the maximizer is unique by 1.30. Fix $(g, \overline{f}) \in \mathscr{F}$, and let $(g_n, \overline{f_n})_{n \in \mathbb{N}}$ be a sequence in G such that

$$\lim_{n\to\infty} ||g_n - g||_q = 0 \text{ and } \lim_{n\to\infty} \rho_Y(\overline{f_n}, \overline{f}) = 0.$$

Now,

$$\begin{aligned}
\left\langle \kappa(g_n, \overline{f_n}), g \right\rangle &= \left\langle \kappa(g_n, \overline{f_n}), g_n \right\rangle + \left\langle \kappa(g_n, \overline{f_n}), g - g_n \right\rangle \\
&= \eta(g_n, \overline{f_n}) + \left\langle \kappa(g_n, \overline{f_n}), g - g_n \right\rangle \\
&\geq \eta(g_n, \overline{f_n}) - ||f_n||_p ||g - g_n||_q.
\end{aligned}$$

Similarly,

$$\left\langle \kappa(g_n, \overline{f_n}), g \right\rangle \leq \eta(g_n, \overline{f_n}) + ||f_n||_p ||g - g_n||_q.$$

Hence, it follows by (i),

$$\lim_{n\to\infty} \left\langle \kappa(g_n, \overline{f_n}), g \right\rangle = \eta(g, \overline{f}).$$

Therefore, by applying Theorem 3.2 if $1 < p < \infty$, or Theorem 3.4 if $p = 1$,

$$\lim_{n\to\infty} ||\kappa(g_n, \overline{f_n}) - \kappa(g, \overline{f})||_p = 0.$$

$\square$

Remark 3.8. If we consider the result for fixed $\overline{f_0}$, then the proof that

$$\eta(g) = \max\left\{\langle f, g\rangle : f \in \overline{f_0}\right\}$$

is Lipschitz continuous is simply:

$$|\eta(g) - \eta(h)| = \left| \int_0^{\omega} f_0^{\triangle} g^{\triangle} \, \mathrm{d}\mu_L - \int_0^{\omega} f_0^{\triangle} h^{\triangle} \, \mathrm{d}\mu_L \right| \leq ||f_0||_p ||g - h||_q.$$

Before I generalise the Mountain Pass Lemma, one more prepatory result is required: Krasnoselski's Theorem, and its applicable corollary.

Definition 3.9 (Carathéodory mapping)**.** Let Ω be an open subset of $\mathbb{R}^N,\ N \in \mathbb{N}$. We say that $g : \Omega \times \mathbb{R}^M \longrightarrow \mathbb{R}$ is a *Carathéodory mapping* if

(i) for all $\xi \in \mathbb{R}^M,\ x \to g(x, \xi)$ is a measurable function,

(ii) for almost all $x \in \Omega,\ \xi \to g(x, \xi)$ is a continuous function.

Proposition 3.10 (Krasnoselski's Theorem). *Let X and Y be two Banach spaces, Ω a Borel subset of $\mathbb{R}^N$, and $g : \Omega \times X \to Y$ a Carathéodory mapping. For each measurable function $u : \Omega \longrightarrow X$, let $G(u)$ be the function from Ω into Y defined by $G(u) = g(x, u(x))$.*
If G maps $L^p(\Omega; X)$ into $L^r(\Omega; Y)$, $1 \leq p, r < \infty$, then G is continuous in the norm topology.

Proof. This is [27, Proposition 1.1 (pg. 77)]. $\square$

Remark 3.11. Note that G need only map from a subspace of $L^p(\Omega; X)$ into a subspace of $L^r(\Omega; Y)$.
For clarity, given a subset Z of a Banach space $(Y, \|\cdot\|_Y)$ and Ω a Borel subset of $\mathbb{R}^N$, where $1 \leq p < \infty$, then $L^p(\Omega, Z)$ is defined as

$$L^p(\Omega; Z) := \left\{ f : \Omega \to Z : \left(\int_\Omega \|f(x)\|_Y^p \ \mathrm{d}x \right)^{\frac{1}{p}} < \infty \right\}.$$

Corollary 3.12. *Let (X, Σ, μ) be a measure interval, let $1 \leq p < \infty$, and let $\tau : \mathbb{R} \to (0, \pi)$ be the function $\tau(s) = \frac{\pi}{2} + \arctan(s)$, with inverse $\tau^{-1} : (0, \pi) \to \mathbb{R}$, $\tau^{-1}(s) = \tan\left(s - \frac{\pi}{2}\right)$. Let $f_0 \in L^p(X; \mathscr{R})$ and $\mathscr{F}$ be the set of rearrangements of f_0 on X. Set*

$$\mathscr{G} = \{\tau \circ f : f \in \mathscr{F}\}$$

where $\mathscr{G} \subset L^p(X; (0, \pi))$. Denote the functions

$$\mathscr{T} : \mathscr{F} \to \mathscr{G},\ \mathscr{T}(u)(x) = \tau(u(x)),$$
$$\mathscr{T}^{-1} : \mathscr{G} \to \mathscr{F},\ \mathscr{T}^{-1}(v)(x) = \tau^{-1}(v(x)).$$

Then $\mathscr{T}$ and $\mathscr{T}^{-1}$ are inverses and both are continuous in the norm topology.

Proof. Note that for every $v \in \mathscr{G}$, we have that it is a measurable function from X to $(0, \pi)$, and so

$$\int_X |v|^p \ \mathrm{d}\mu \leq \pi^p \mu(X) < \infty,$$

so it follows $v \in L^p(X; (0, \pi))$.
First, we need to show that $\mathscr{T}$ and $\mathscr{T}^{-1}$ are inverses. Suppose $u \in \mathscr{F}$, $v \in \mathscr{G}$ and let $x \in X$ be arbitrary. Then

$$\left(\mathscr{T}^{-1} \circ \mathscr{T}\right)(u)(x) = \tau^{-1}\left(\mathscr{T}(u)(x)\right) = \tau^{-1}\left(\tau(u(x))\right) = u(x),$$
$$\left(\mathscr{T} \circ \mathscr{T}^{-1}\right)(v)(x) = \tau\left(\mathscr{T}(v)(x)\right) = \tau\left(\tau^{-1}(v(x))\right) = v(x).$$

Hence, $\mathscr{T}$ and $\mathscr{T}^{-1}$ are inverses.

(X, Σ, μ) is a measure interval, and so we may consider it to be a Borel subset of $\mathbb{R}$. Let

$$g : X \times \mathbb{R} \to (0, \pi),\ g(x, y) = \tau(y).$$

Then as τ is continuous, it means that g is a Carathéodory mapping. The function $\mathscr{T}$ now satisfies the conditions of Proposition 3.10, so $\mathscr{T}$ is continuous in the norm topology.

To prove the continuity of $\mathscr{T}^{-1}$, we follow the proof of [16, Lemma 2.10]. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{G}$ converging to the limit $v \in \mathscr{G}$ in the norm topology, and let $\varepsilon > 0$. Let $M \in \mathbb{N}$, and define

$$\gamma_M : (0, \pi) \to \mathbb{R},\ \gamma_M(s) = \begin{cases} M & \text{if } \tau^{-1}(s) \geq M, \\ \tau^{-1}(s) & \text{if } -M \leq \tau^{-1}(s) \leq M, \\ -M & \text{if } \tau^{-1}(s) \leq -M \end{cases}$$

$$\theta_M(s) = \tau^{-1}(s) - \gamma_M(s).$$

Fix $M \in \mathbb{N}$ sufficiently large that $||\theta_M \circ v||_p < \frac{\varepsilon}{5}$, so $||\theta_M \circ v_n||_p < \frac{\varepsilon}{5}$ for all $n \in \mathbb{N}$, by the equimeasurability of rearrangements. Then

$$\begin{aligned} ||\mathscr{T}^{-1}(v_n) - \mathscr{T}^{-1}(v)||_p &\leq ||\gamma_M \circ v_n - \gamma_M \circ v||_p + ||\theta_M \circ v_n||_p + ||\theta_M \circ v||_p \\ &< ||\gamma_M \circ v_n - \gamma_M \circ v||_p + \frac{2\varepsilon}{5}. \end{aligned}$$

Now, ρ_M is bounded and continuous, meaning that by the Dominated Convergence Theorem, there exists $N \in \mathbb{N}$ sufficiently large such that

$$n > N \text{ implies that } ||\gamma_M \circ v_n - \gamma_M \circ v||_p < \frac{3\varepsilon}{5}.$$

Then,

$$n > N \text{ implies that } ||\mathscr{T}^{-1}(v_n) - \mathscr{T}^{-1}(v)||_p < \varepsilon.$$

Hence, $\mathscr{T}^{-1}$ is continuous in the norm topology. □

Now, I show two paths in the set of rearrangements on a measure interval may be continuously deformed into one another.

Lemma 3.13. *Let (X, Σ, μ) be a measure interval, let $1 \leq p < \infty$, let q be the conjugate exponent of p, let $f_* \in L^p(\mu)$, let $\mathscr{F}$ be the set of rearrangements of f_* on X. Suppose $(g_t)_{t\in[0,1]} \subseteq L^q(X)$ is a path such that every level set of g_t has zero measure for each $t \in [0, 1]$, and let φ_t be an increasing function such that $f_t^\dagger = \varphi_t \circ g_t \in \mathscr{F}$. Let $f_t \in \mathscr{F}$*

such that $(f_t)_{t\in[0,1]} \subseteq L^p(X)$ *is a path.*
Then there is a continuous function $\psi : [0,1]\times[0,1] \to \mathscr{F}$ *such that* $\psi(0,t) = f_t$ *and* $\psi(1,t) = f_t^\dagger$*, along which* $\langle \psi(s,u), g_u\rangle$ *is an increasing function of* $0 \le s \le 1$ *for each* $s \in [0,1]$.

Proof. We assume without loss of generality that $X = [0,\omega]$ equipped with the Lebesgue measure, as (X,Σ,μ) is a measure interval. Define $\tau : \mathbb{R} \to (0,\pi)$ by $\tau(s) = \frac{\pi}{2}+\arctan(s)$, with inverse $\tau^{-1} : (0,\pi) \to \mathbb{R}$, $\tau^{-1}(s) = \tan\left(s - \frac{\pi}{2}\right)$. Set $\mathscr{G} = \{\tau\circ f : f \in \mathscr{F}\} \subset L^p(X;(0,\pi))$, and

$$\begin{aligned}\mathscr{T} : \mathscr{F} \to \mathscr{G},\ \mathscr{T}(u)(x) = \tau(u(x)),\\ \mathscr{T}^{-1} : \mathscr{G} \to \mathscr{F},\ \mathscr{T}^{-1}(v)(x) = \tau^{-1}(v(x)).\end{aligned}$$

Let $F_t = \mathscr{T}(f_t)$ and let $F_t^\dagger = \mathscr{T}(f_t^\dagger)$, so $F_t, F_t^\dagger \in \mathscr{G} \subset L^p(X;(0,\pi))$. For each $-\infty \le s \le \infty, 0 \le t \le 1$ and $x \in X$, define

$$\begin{aligned}\Lambda_t(s) &= \{y \in \Omega : g_t(y) \ge s\}\\ \Gamma(s,t) &= F_t \mathbb{1}_{\Lambda_t(s)}\\ H(s,t) &= \Gamma(s,t) - F_t.\end{aligned}$$

Next, since the level sets of g_t are of zero measure, choose an increasing function $\varphi_{(s,t)}$ such that $\varphi_{(s,t)}\circ g_t|_{\Lambda_t(s)}$ is a rearrangement of $F_t|_{\Lambda_t(s)}$. Extend $\varphi_{(s,t)}$ so that $\varphi_{(s,t)}(r) = 0$ for $r < s$. Define $\Gamma^\wedge(s,t) = \varphi_{(s,t)}\circ g_t$ on X and define

$$\Gamma^\dagger(s,t) = \Gamma^\wedge(s,t) + H(s,t).$$

Notice that $\Gamma^\dagger(-\infty,t) = F_t^\dagger$, $\Gamma^\dagger(\infty,t) = F_t$ and $\Gamma^\dagger(s,t)$ is a rearrangement of F_t for each $s \in [-\infty,\infty]$, by an application of Proposition 1.20 (*xii*). We seek to demonstrate the continuity of $\Gamma^\dagger, \Gamma : [-\infty,\infty]\times[0,1] \to \mathscr{F}$.
Fix $(s,t) \in (-\infty,\infty)\times[0,1]$, and let $(s',t') \in (-\infty,\infty)\times[0,1]$. Consider that

$$\begin{aligned}\left\|\Gamma^\dagger(s,t) - \Gamma^\dagger(s',t')\right\|_p &\le \left\|\Gamma^\wedge(s,t) - \Gamma^\wedge(s',t')\right\|_p + \left\|H(s,t) - H(s',t')\right\|_p\\ &\le \left\|\Gamma^\wedge(s,t) - \Gamma^\wedge(s',t')\right\|_p + \left\|\Gamma(s,t) - \Gamma(s',t')\right\|_p + \|F_t - F_{t'}\|_p.\end{aligned}$$

As $(F_t)_{t\in[0,1]}$ is a path, the continuity of $\Gamma^\dagger$ follows from the continuity of Γ and $\Gamma^\wedge$. I shall demonstrate the continuity of Γ first. By construction,

$$|F_t(x)| < \pi \text{ for all } x \in X \text{ and } t \in [0,1].$$

Hence,

$$\begin{aligned}
||\Gamma(s,t) - \Gamma(s',t')||_p^p &\leq \int_X \left|F_t \mathbb{1}_{\Lambda_t(s)} - F_{t'} \mathbb{1}_{\Lambda_{t'}(s')}\right|^p \, \mathrm{d}\mu \\
&\leq \|F_t - F_{t'}\|_p^p + \int_{\Lambda_t(s)\setminus\Lambda_{t'}(s')} |F_t|^p \, \mathrm{d}\mu + \int_{\Lambda_{t'}(t)\setminus\Lambda_t(s)} |F_{t'}|^p \, \mathrm{d}\mu \\
&\leq \|F_t - F_{t'}\|_p^p + \pi^p \left(\mu\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) + \mu\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right)\right)
\end{aligned}$$

Now, it is required to show that

$$\lim_{(s',t')\to(s,t)} \mu\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) = \lim_{(s',t')\to(s,t)} \mu\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) = 0.$$

I seek to show that

$$\lim_{(s',t')\to(s,t)} \mu\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) = 0.$$

Fix $(s,t) \in [-\infty,\infty] \times [0,1]$, and let $(s',t') \in [-\infty,\infty] \times [0,1]$. Note that

$$\Lambda_t(s) \setminus \Lambda_{t'}(s') = g_t^{-1}[s,\infty] \cap g_{t'}^{-1}[-\infty,s').$$

Assume, without loss of generality, that $s, s' \in \mathbb{R}$. For the case $s' = -\infty$, one needs to observe that $\mu\left(g_t^{-1}(\infty)\right) \to 0$ as $t \to t'$. As $(g_t)_{t\in[0,1]}$ is a path in $L^q(X)$, then the functions also converge in the measure μ. Define for every $\varepsilon > 0$

$$\mathscr{E}(t,t';\ \varepsilon) := \{y \in X : |g_t(y) - g_{t'}(y)| \leq \varepsilon\}.$$

For each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that

$$t' \in [0,1] \cap (t-\delta, t+\delta) \text{ implies that } \mu\left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right) < \frac{\eta}{2}.$$

Let $y \in \mathscr{E}(t,t';\ \varepsilon) \setminus \Lambda_{t'}(s')$, so $g_t(y) < g_{t'}(y) + \varepsilon < s' + \varepsilon$, so

$$\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) \cap \mathscr{E}(t,t';\ \varepsilon) = g_t^{-1}[s,\infty] \cap g_{t'}^{-1}[-\infty,s') \cap \mathscr{E}(t,t';\ \varepsilon) \subseteq g_t^{-1}[s,s'+\varepsilon).$$

Choose $\varepsilon > 0$ sufficiently small such that, for all $s' \in (s-\varepsilon, s+\varepsilon)$,

$$\mu\left(g_t^{-1}[s,s'+\varepsilon)\right) < \frac{\eta}{2}.$$

Then for $(s', t') \in (s-\varepsilon, s+\varepsilon) \times ([0,1] \cap (t-\delta, t+\delta))$, it follows

$$\begin{aligned}\Lambda_t(s) \setminus \Lambda_{t'}(s') &= \left(\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) \cap \mathscr{E}(t,t';\ \varepsilon)\right) \cup \left(\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) \setminus \mathscr{E}(t,t';\ \varepsilon)\right) \\ &\subseteq g_t^{-1}[s, s'+\varepsilon) \cup \left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right),\end{aligned}$$

which implies that

$$\mu\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) \leq \mu\left(g_t^{-1}[s, s'+\varepsilon)\right) + \mu\left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right) < \frac{\eta}{2} + \frac{\eta}{2} = \eta.$$

Hence, the conclusion is that

$$\lim_{(s',t') \to (s,t)} \mu\left(\Lambda_t(s) \setminus \Lambda_{t'}(s')\right) = 0.$$

The justification of the second limit is to be elaborated. I seek to show the limit

$$\lim_{(s',t') \to (s,t)} \mu\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) = 0.$$

Fix $(s,t) \in (-\infty, \infty) \times [0,1]$, and let $(s', t') \in (-\infty, \infty) \times [0,1]$. It follows that

$$\begin{aligned}\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) \cap \mathscr{E}(t,t';\ \varepsilon) &= g_{t'}^{-1}[s', \infty] \cap g_t^{-1}[-\infty, s) \cap \mathscr{E}(t,t';\ \varepsilon) \\ &\subseteq g_t^{-1}[s'-\varepsilon, \infty] \cap g_t^{-1}[-\infty, s) = g_t^{-1}[s'-\varepsilon, s).\end{aligned}$$

Choose $\varepsilon, \delta > 0$ sufficiently small such that, for all $(s', t') \in (s-\varepsilon, s+\varepsilon) \times [0,1] \cap (t-\delta, t+\delta)$, we have

$$\mu\left(g_t^{-1}[s'-\varepsilon, s)\right) < \frac{1}{2}\eta \text{ and } \mu\left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right) < \frac{1}{2}\eta.$$

Then

$$\begin{aligned}\Lambda_{t'}(s') \setminus \Lambda_t(s) &= \left(\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) \cap \mathscr{E}(t,t';\ \varepsilon)\right) \cup \left(\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) \setminus \mathscr{E}(t,t';\ \varepsilon)\right) \\ &\subseteq g_t^{-1}[s'-\varepsilon, s) \cup \left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right).\end{aligned}$$

Thus, for all $(s', t') \in (s-\varepsilon, s+\varepsilon) \times [0,1] \cap (t-\delta, t+\delta)$, it follows

$$\begin{aligned}\mu\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) &\leq \mu\left(g_t^{-1}[s'-\varepsilon, s)\right) + \mu\left(X \setminus \mathscr{E}(t,t';\ \varepsilon)\right) \\ &< \frac{1}{2}\eta + \frac{1}{2}\eta \\ &= \eta.\end{aligned}$$

Thus, we conclude that

$$\lim_{(s',t')\to(s,t)} \mu\left(\Lambda_{t'}(s') \setminus \Lambda_t(s)\right) = 0.$$

The continuity of $\Gamma^\wedge$ is a corollary of the continuity of Γ and the application of Lemma 3.7 (ii). Therefore, $\Gamma^\dagger : [-\infty,\infty] \times [0,1] \to \mathscr{G}$ is continuous. Now, define

$$\psi : [-\infty,\infty] \times [0,1] \to \mathscr{F},\ \psi = \mathscr{T}^{-1}\left(\Gamma^\dagger\right).$$

By Corollary 3.12, $\mathscr{T}^{-1}$ is continuous, and so ψ is continuous. For each $t \in [0,1]$, $\psi(\cdot,t)$ forms a continuous path from $f_t^\dagger$ to f_t. Furthermore, we seek to show that $\langle\psi(\cdot,t), g_t\rangle$ is a decreasing function of $s \in [-\infty,\infty]$. Let $-\infty < s \leq s' < \infty$. Then

$$\left\langle\psi(s,t) - \psi(s',t), g_t\right\rangle = \int_{\Lambda_t(s)} \psi(s,t) g_t \,\mathrm{d}\mu - \left(\int_{\Lambda_t(s')} \psi(s',t) g_t \,\mathrm{d}\mu + \int_{\Lambda_t(s)\setminus\Lambda_t(s')} f_t g_t \,\mathrm{d}\mu\right).$$

Relative to $\Lambda_t(s)$, the equality ensures that $\psi(s,t) = \tau^{-1} \circ \varphi_{(s,t)} \circ g_t$, and the function $\psi(s,t)$ is a rearrangement of $\psi(s',t) + \mathbb{1}_{\Lambda_t(s)\setminus\Lambda_t(s')} f_t$. By [16, Lemma 2.4], it is immediate that

$$\left\langle\psi(s,t) - \psi(s',t)\right\rangle \geq 0$$

as required. Rescaling and reversing the first variable gives the result. □

Corollary 3.14. *Let (X,Σ,μ) be a measure interval, let $1 \leq p < \infty$, let $f_* \in L^p(X)$. Let $\mathscr{F}$ be the set of rearrangements of f_* on X. Suppose $(f_t)_{t\in[0,1]},\ (g_t)_{t\in[0,1]} \subseteq L^p(X)$ are two paths in $\mathscr{F}$. Then there is a continuous function $\psi : [0,1]\times[0,1] \to \mathscr{F}$ such that $\psi(0,t) = f_t$ and $\psi(1,t) = g_t$ for each $t \in [0,1]$.*

Proof. Let q be conjugate exponent of p. Fix $(h_t)_{t\in[0,1]} \subset L^q(X)$, where every level set of h_t has zero measure, and let φ_t be an increasing function such that $f_t^* = \varphi_t \circ h_t \in \mathscr{F}$ for each $t \in [0,1]$. Then there are two continuous functions $\psi_1, \psi_2 : [0,1]\times[0,1] \to \mathscr{F}$ such that

$$\psi_1(0,u) = f_u \text{ and } \psi_1(1,u) = f_u^*,$$
$$\psi_2(0,u) = g_u \text{ and } \psi_2(1,u) = f_u^*.$$

Set $\psi : [0,1]\times[0,1] \to \mathscr{F}$ by

$$\psi(s,t) = \begin{cases} \psi_1(2s,t) & 0 \leq s \leq \frac{1}{2}, \\ \psi_2(2-2s,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then ψ satisfies the conditions and is continuous. $\square$

3.2 The Generalised Mountain Pass Lemma

Finally, I state and prove the main result of this subsection: the Mountain Pass Lemma over discs, rather than intervals.

Theorem 3.15. *Let (X, Σ, μ) be a measure interval, let $1 \leq p < \infty$, and let $\Psi : L^p(X) \to \mathbb{R}$ be a continuously differentiable convex functional, let $f_0 \in L^p(X)$. Denote $\mathscr{F}$ be the set of rearrangements of f_0 on X. Denote the closed unit disc in $\mathbb{R}^2$ by B. Suppose $\gamma : \partial B \to \mathscr{F}$ is continuous, and $\gamma(\partial B)$ is a simple and closed loop in $\mathscr{F}$. Define*

$$\mathscr{C} = \{h \in C(B, \mathscr{F}) : h|_{\partial B} = \gamma\}$$
$$c = \sup_{h \in \mathscr{C}} \inf_{x \in D} \Psi(h(x))$$

Suppose

$$\inf \Psi(L^p(\mu)) < c < \inf \Psi(\gamma).$$

Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ satisfying

$$\lim_{n\to\infty} \Psi(v_n) = c,$$
$$\lim_{n\to\infty} \left(\sup_{\mathscr{F}} \big\langle \,\cdot\, , \Psi'(v_n)\big\rangle - \big\langle v_n, \Psi'(v_n)\big\rangle \right) = 0.$$

Proof. This proof will extend the proof given in [16]. As (X, Σ, μ) is a measure interval, it may be assumed without loss of generality that $X = [0, \omega]$ equipped with the Lebesgue measure μ_L. Denote $\eta(v) = \sup_{f\in\mathscr{F}} \langle f, v\rangle$, where $\langle f, g\rangle = \int_0^\omega fg \, \mathrm{d}\mu_L$.

Let $\varepsilon > 0$ satisfying $c + 2\varepsilon < \inf \Psi(\gamma)$ and $c - 2\varepsilon > \inf \Psi(L^p[0, \omega])$. I will prove the existence of a point $v \in \mathscr{F}$ satisfying

$$c - 2\varepsilon < \Psi(v) < c + 2\varepsilon,$$
$$\eta(\Psi'(v)) - \big\langle v, \Psi'(v)\big\rangle < 2\varepsilon.$$

Assume, seeking contradiction, that no point in the set $\mathscr{F}$ satisfies both inequalities. Consider any $h \in \mathscr{C}$ satisfying $\inf \Psi(h(B)) > c - \varepsilon$. Let V_m denote the linear subspace spanned by $\sin\left(\frac{\pi x}{\omega}\right), \ldots, \sin\left(\frac{m\pi x}{\omega}\right)$. Let $(\zeta_m)_{m\in\mathbb{N}}$ be a sequence of nearest point maps to $(V_m)_{m\in\mathbb{N}}$, then all such elements of this sequence are strongly continuous. Define

$$\beta = \inf \big\{ ||\Psi'(h(x))||_q : x \in B \big\},$$

which means that $\beta > 0$. This is true because Ψ is uniform,

$$c - 2\varepsilon > \inf \Psi(L^p[0,\omega]) \text{ and } \inf \Psi(h([0,1])) > c - \varepsilon.$$

Choose $m \in \mathbb{N}$ sufficiently large such that

$$||\Psi'(h(x)) - \zeta_m \Psi'(h(x))||_q < \delta = \min\left\{\beta, \frac{\varepsilon}{2||f_0||_p}\right\}$$

for all $x \in B$. This is uniform within in B because ζ_m is strongly continuous. In particular, $\zeta_m \Psi'(h(x)) \neq 0$ for all $x \in B$.

Since each level set of $\zeta_M \Psi'(h(x))$ is finite, they each have zero measure. For each $x \in B$, the functional $\langle \,\cdot\, , \zeta_m \Psi'(h(x))\rangle$ has a unique maximiser relative to $\mathscr{F}$, by Lemma 1.30, which we call $\kappa(x)$. Now, it follows $\kappa(x) = \varphi_x \circ (\zeta_m \Psi'(h(x)))$ almost everywhere for some increasing function $\varphi_x : \mathbb{R} \to \mathbb{R}$. Applying Lemma 3.7 yields that $\kappa : D \to \mathscr{F}$ is continuous.

I now seek to show that $\Psi(\kappa(x)) > c$. Suppose $\Psi(h(x)) \geq c + 2\varepsilon$. Then

$$\begin{aligned}
\Psi(\kappa(x)) &\geq \Psi(h(x)) + \big\langle \kappa(x) - h(x), \Psi'(h(x))\big\rangle && \text{by convexity of } \Psi \\
&\geq \Psi(h(x)) + \big\langle \kappa(x) - h(x), \zeta_m \Psi'(h(x))\big\rangle - 2||f_0||_p \delta && \text{by prior calculation} \\
&\geq \Psi(h(x)) - 2||f_0||_p \delta && \text{by definition of } \kappa(x) \\
&\geq c + 2\varepsilon - \varepsilon && \text{by prior calculation} \\
&= c + \varepsilon.
\end{aligned}$$

Suppose $\Psi(h(x)) < c + 2\varepsilon$; by choice of h we also have $c - 2\varepsilon < \Psi(h(x))$, so it follows that, by utilising the assumption that $\inf \Psi(h[0,1]) > c - \varepsilon$,

$$\eta(\Psi'(h(x))) \geq \big\langle h(x), \Psi'(h(x))\big\rangle + 2\varepsilon.$$

Thus,

$$\begin{aligned}
\Psi(\kappa(x)) &\geq \Psi(h(x)) + \big\langle \kappa(x) - h(x), \Psi'(h(x))\big\rangle && \text{by convexity of } \Psi \\
&\geq \Psi(h(x)) + \big\langle \kappa(x), \zeta_m \Psi'(h(x))\big\rangle - \delta||f_0||_p - \big\langle h(x), \Psi' h(x)\big\rangle && \text{by choice of } m \\
&\geq \Psi(h(x)) + \eta(\zeta_m \Psi'(h(x))) - \eta(\Psi'(h(x))) - \delta||f_0||_p + 2\varepsilon && \text{by Lemma 3.7} \\
&\geq \Psi(h(x)) - 2\delta||f_0||_p + 2\varepsilon && \text{by prior calculation} \\
&\geq c + \varepsilon.
\end{aligned}$$

This proves that $\Psi(\kappa(x)) \geq c + \varepsilon > c$.

Let $\hat{B} = \left\{x \in \mathbb{R}^2 : \frac{1}{2} \leq ||x|| \leq 1\right\}$ be an annulus in $\mathbb{R}^2$. By Lemma 3.13, there exists a continuous map $\hat{\kappa}$ in $C(\hat{B}, \mathscr{F})$ that the following properties: I shall treat elements of $\hat{B}$ as if they were elements of $S^1 \times [\frac{1}{2}, 1]$, $\langle \hat{\kappa}(\vartheta, r), \zeta_m \Psi'(\hat{\kappa}(\vartheta, 1)) \rangle$ is a decreasing function of $\frac{1}{2} \leq r \leq 1$; and

$$\hat{\kappa}(y) = \kappa(y) \text{ for all } y \in \hat{D} \text{ with } ||y|| = \frac{1}{2}$$
$$\hat{\kappa}(y) = \gamma(y) \text{ for all } y \in \hat{D} \text{ with } ||y|| = 1.$$

Denote $\hat{x} = \frac{x}{||x||}$, and by applying the previous calculations, it is yielded that:

$$\begin{aligned}
\Psi(\hat{\kappa}(x)) &\geq \Psi(\hat{\kappa}\left(\hat{x}\right)) + \left\langle \hat{\kappa}(x) - \hat{\kappa}(\hat{x}), \Psi'(\hat{\kappa}(\hat{x})) \right\rangle \\
&\geq \Psi(\gamma(\hat{x})) + \left\langle \hat{\kappa}(x) - \hat{\kappa}(\hat{x}), \zeta_m \Psi'(\hat{\kappa}(\hat{x})) \right\rangle - 2\delta ||f_0||_p \\
&\geq (c + 2\varepsilon) + 0 - \varepsilon \\
&= c + \varepsilon.
\end{aligned}$$

Define

$$h_0(x) = \kappa(x), \; 0 \leq ||x|| \leq \frac{1}{2},$$
$$h_0(x) = \hat{\kappa}(x), \; \frac{1}{2} \leq ||x|| \leq 1.$$

This is then a continuous function from D to $\mathscr{F}$ that satisfies $h_0|_{\partial D} = \gamma$, that is, $h_0 \in \mathscr{C}$. Then $h_0 \in \mathscr{C}$ and $\Psi(h_0(x)) > c$ for $x \in D$ contrary to the definition of c. For all sufficiently small $\varepsilon > 0$, we can choose $v \in \mathscr{F}$ satisfying both of the inequalities. $\square$

This means that corollaries to the Mountain Pass Lemma follow too. Here is the corollary given in [16], which follows a similar proof.

Corollary 3.16. *Let $\Omega \subseteq \mathbb{R}^N$ be an open and bounded set, equipped with the N-dimensional Lebesgue measure μ (or another equivalent measure), let $1 < p < \infty$, let $p^{-1} + q^{-1} = 1$, let*

$$\mathscr{L}(u) = \sum_{1 \leq |\alpha| \leq m} a^\alpha D^\alpha(u)$$

define a linear partial differential operator on Ω, where the a^α are measurable functions for $1 \leq |\alpha| \leq m$, and there is no 0-th order term. Let $K : L^p(\Omega) \to L^q(\Omega)$ be a compact, symmetric, positive linear operator, suppose $Kv \in W^m(\Omega)$ and $\mathscr{L}Kv = v$ almost everywhere in Ω for all $v \in L^p(\Omega)$ and let $w \in L^q(\Omega) \cap W^m(\Omega)$ satisfy $\mathscr{L}w = 0$

almost everywhere in Ω. *Let*

$$\Psi(v) = \frac{1}{2}\int_\Omega vKv \; d\mu + \int_\Omega vw \; d\mu$$

for all $v \in L^p(\Omega)$, *let* $f_0 \in L^p(\Omega)$ *be non-negative, let* $\mathscr{F}$ *be the set of rearrangements of* f_0 *on* Ω. *Suppose* $\gamma : \partial B \to \mathscr{F}$ *is continuous, and* $\gamma(\partial B)$ *is a closed and simple loop in* $\mathscr{F}$, *where* B *denotes the unit disc in* $\mathbb{R}^2$ *and define*

$$\begin{aligned} \mathscr{C} &= \{h \in C(B, \mathscr{F}) : h|_{\partial B} = \gamma\} \\ c &= \sup_{h\in\mathscr{C}} \inf_{x\in B} \Psi(h(x)). \end{aligned}$$

Suppose

$$\inf \Psi(L^p(\mu)) < c < \inf \Psi(\gamma).$$

Then there exists $v \in \mathscr{F}$ *and* $u = Kv + w$ *such that*

$$\begin{aligned} \Psi(v) &= c, \\ \mathscr{L}u &= \varphi \circ u \end{aligned}$$

almost everywhere in Ω, *for some increasing function* $\varphi : \mathbb{R} \to \mathbb{R}$.

Proof. Denote

$$\eta : \mathscr{F} \to \mathbb{R}, \;\; \eta(g) = \sup_{\mathscr{F}} \langle \, \cdot \, , g\rangle$$

Thus, we may derive the existence of a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ such that

$$\begin{aligned} \lim_{n\to\infty} \Psi(v_n) &= c \\ \lim_{n\to\infty} \eta(Kv_n + w) - \langle v_n, Kv_n + w\rangle &= 0. \end{aligned}$$

By passing to a subsequence, we may assume without loss of generality that $v_n \rightharpoonup v$ weakly in $L^p(\Omega)$, for some $v \in \overline{\mathscr{F}^w}$. Then, by the compactness of K, $\lim_{n\to\infty} Kv_n = Kv$ in the q-norm. It follows that $\lim_{n\to\infty} \eta(Kv_n + w) = \eta(Kv + w)$. Hence,

$$\langle v, Kv + w\rangle = \eta(Kv + w).$$

Also, the equation $\mathscr{L}(Kv + w) = 0$ almost everywhere. By Lemma 1.30, there must be exist an increasing $\varphi : \mathbb{R} \to \mathbb{R}$ such that $v = \varphi \circ (Kv + w) = \mathscr{L}(Kv + w)$. Also, $\Psi(v) = c$.

Setting $u = Kv + w$ means that u has the desired properties. □

Definition 3.17 (Spanning Surface)**.** Let $B \subseteq \mathbb{R}^2$ be the unit disc in $\mathbb{R}^2$, equipped with the Euclidean metric. Let Y be a topological space, then a continuous function $\Gamma : B \to Y$ is called a *spanning surface of* $\Gamma(\partial B)$.

Remark 3.18. There are multiple ways of describing spanning surfaces. The unit disc may be characterised in Euclidean or polar co-ordinates, or a spanning curve is a continuous function $\Gamma : [0,1] \times [0,1] \to X$ which satisfies

$$\Gamma(s,i) = \Gamma(s',i) \text{ for all } s, s' \in [0,1] \text{ and } i \in \{0,1\}\,.$$

Corollary 3.19. *Let (X, Σ, μ) be a measure interval, let $1 < p < \infty$, and let $f_* \in L^p(X)$. Denote by $\mathscr{F}$ the set of rearrangements of f_* on X. Let $\Psi : L^p(X) \to \mathbb{R}$ be a continuously differentiable convex function, and let $\gamma : \partial B \to \mathscr{F}$ be continuous, where $\gamma(\partial B)$ is a simple and closed loop in $\mathscr{F}$, and fix $f_0 \in \mathscr{F}$, and let B denote the closed unit disc in $\mathbb{R}^2$. Define $f_1 \in \gamma\,(\partial B)$ to satisfy $\Psi(f_1) = \inf \Psi(\gamma)$ and*

$$\begin{aligned}
\mathscr{C}_0 &= \{h \in C(B, \mathscr{F}) : h(0) = f_0,\ h|_{\partial B} = \gamma\}\,,\\
\mathscr{C}_1 &= \{h \in C([0,1], \mathscr{F}) : h(0) = f_0,\ h(1) = f_1\}\,,\\
c_0 &= \sup_{h \in \mathscr{C}_0} \inf_{x \in D} \Psi(h(x)),\\
c_1 &= \sup_{h \in \mathscr{C}_1} \inf_{0 \le t \le 1} \Psi(h(t)).
\end{aligned}$$

Then

$$c_0 < \Psi(f_1) \Longleftrightarrow c_1 < \Psi(f_1).$$

Proof. The contrapositive of the corollary is that there is a path where every point has values of Ψ larger than or equal to $\Psi(f_1)$ if, and only if, there is a spanning curve where every point has values of Ψ larger than or equal to $\Psi(f_1)$. Suppose there is a path $h : [0,1] \to \mathscr{F}$ from f_0 to f_1 such that $\Psi(h(t)) \ge \Psi(f_1)$ for all $t \in [0,1]$. Now, using part of the proof of the Mountain Pass Lemma yields a spanning curve $\Gamma_1 : [0,1] \times [0,1] \to \mathscr{F}$ that satisfies $\Gamma_1(t,0) = f_1,\ \Gamma_1(1,x) = \gamma(x)$ and $\Psi(\Gamma_1(t,x)) \ge \Psi(f_1)$ for all $t, x \in [0,1]$. We can define $\Gamma : [0,1] \times [0,1] \to \mathscr{F}$ by

$$\begin{aligned}
\Gamma(t,x) &= h(2t) \text{ for all } 0 \le t \le \frac{1}{2},\ x \in [0,1],\\
\Gamma(t,x) &= \Gamma_1(2t-1, x) \text{ for all } \frac{1}{2} \le t \le 1,\ x \in [0,1].
\end{aligned}$$

Now, Γ is a spanning curve of $\mathscr{F}$ that has every point with Ψ-value greater than $\Psi(f_1)$. Conversely, suppose there is a spanning curve Γ with centre f_0 and boundary γ such that $\Psi(\Gamma(y)) \ge \Psi(f_1)$ for all $y \in B$. Set h to be the line from the centre of B to f_1, then this is a path such that $\Psi(h(t)) \geq \Psi(f_1)$ for all $t \in [0,1]$. $\square$

3.3 Other Generalisations and Applications

The circular Mountain Pass Lemma has numerous applications. It can replicate the results of the linear Mountain Pass Lemma, but also provide some further results. The former case is achieved by simply setting the set of continuous functions from the unit disc B to $\mathscr{F}$ to be equal to two specified relative maxima – say $v_1, v_2 \in \mathscr{F}$ – at two distinct points on the boundary of the unit disc, that is,

$$\mathscr{C} = \{h \in C(B, \mathscr{F}) : h(1,0) = v_1 \text{ and } h(-1,0) = v_2\}.$$

There are other methods of generalising the Mountain Pass Lemma, such as for metric and Banach spaces, as in [44, Chapter 16]. The consideration of Palais-Smale conditions has also been made by [5], with an application to elliptic partial differential equations. The paper concerns the existence and estimates for the number of critical points possessed by a real-valued continuously differentiable functional on a real Banach space, using arguments from mountain passes.

Definition 3.20 (δ-regular, δ-regularity mapping, point-regular, critical point, δ-regularity constant)**.** Let (Y, ρ_Y) be a metric space, $y \in Y$, and f to be a real function defined in a neighbourhood of y and $\delta > 0$. The point y is called *δ-regular* if there is a neighbourhood U of y, a constant $\alpha > 0$, and a continuous mapping $\zeta : U \times [0,\alpha] \to Y$ such that for all $(u,t) \in U \times [0,\alpha]$:

(i) $\operatorname{dist}(\zeta(u,t), u) \leq t$;

(ii) $f(u) - f(\zeta(u,t)) \geq \delta t$.

The function ζ is termed a *δ-regularity mapping* for f at y. Concordantly, y is labelled *point-regular* if there exists a $\delta > 0$ such that y is δ-regular. Otherwise, y is a *critical point* of f.

The *δ-regularity constant* of f at a point-regular y is

$$\delta(f,y) = \sup\{\delta > 0 : f \text{ is } \delta\text{-regular at } y\}.$$

If y is a critical point of f, then set $\delta(f,y) = 0$.

The previous forms of the Mountain Pass Lemma have not used the Palais-Smale conditions. These conditions can be used for continuous functions defined on metric spaces.

Definition 3.21 (Palais-Smale condition)**.** Suppose (Y, ρ_Y) is a metric space, and $f : Y \to \mathbb{R}$ is continuous. It is said f satisfies the *Palais-Smale conditions* at the level $c \in \mathbb{R}$ if:
any sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ such that $\lim_{n\to\infty} f(y_n) = c$ and $\lim_{n\to\infty} \delta(f, y_n) = 0$ has a convergent subsequence.

A metric formulation of the Mountain Pass Lemma was given by Degiovanni and Marzocchi in [22]. I provide the sup-min form, rather than the inf-max form, as they are plainly equivalent.

Theorem 3.22. *Let (Y, ρ_Y) be a metric space, $f : Y \to \mathbb{R}$ be a continuous function. Suppose D is compact, and $S \subseteq D$ is a closed subset, $\psi : S \to Y$ be continuous and*

$$\Gamma = \{\gamma \in C(D; Y) : \gamma|_S = \psi\}.$$

Suppose that Γ is non-empty, and

$$\text{for all } \gamma \in \Gamma, \ \inf_{\gamma(D)} f < \inf_{\psi(S)} f.$$

If f satisfies the Palais-Smale condition at the level

$$c = \sup_{\gamma\in\Gamma} \inf_{\gamma(D)} f,$$

then c is a critical value of f.

Proof. This is [44, Theorem 16.7]. □

Remark 3.23. (i) The role of the Palais-Smale conditions in the proof of this result is to prove the existence of a convergent subsequence at level c, that is, it is a compactness condition on the function, as seen in [44, Remark 2.3].

(ii) It should be noted what the name of this result, the Mountain Pass Lemma, means. The Mountain Pass Lemma can be considered to be a more general form of the Mean Value Theorem.

Let Y be a metric space of ordered pairs of planetary co-ordinates and elevation. Suppose D is some compact set of co-ordinates, like a region around a mountain, and S refers to a closed subset. Allow me to assume that the S refers to two points, representing peaks. Suppose the co-ordinates vary continuously with the elevation, so f becomes the projection function that produces the elevation as an output. The set Γ is the set of all continuous paths where these paths agree with

ψ on S. If S is two points, all paths must agree with the value of ψ at these two points. The requirement that

$$\text{for all } \gamma \in \Gamma,\ \inf_{\gamma(D)} f < \inf_{\psi(S)} f$$

means these two points must represent peaks, so all of the paths in Γ dip beneath these local peaks in terms of their values of f, their elevation. These paths can be thought of as passes through mountains, so there must be a saddle point in the mountain region. A detailed discussion of the Mountain Pass Lemma may be read

Figure 3-1: This is a photograph of the Cascade Pass in Washington, U.S.

in [44].

Now, we should apply this result.

Let $\Omega \subset \mathbb{R}^2$ be the unit disc in $\mathbb{R}^2$ with measure μ, let $1 < p, q < \infty$ be conjugates and suppose $K : L^p(\Omega) \to L^q(\Omega)$ is the linear inverse of the negative Laplacian. Set, for some $\alpha > 0$,

$$\Psi_\alpha : L^p(\Omega) \to \mathbb{R},\ \Psi_\alpha(u) = \frac{1}{2}\int_\Omega uKu \ \mathrm{d}\mu + \alpha\left(\int_\Omega x_1 u \ \mathrm{d}\mu\right)^2 + \alpha\left(\int_\Omega x_2 u \ \mathrm{d}\mu\right)^2.$$

This energy functional is convex, as can be shown for $u, v \in L^p(\Omega), t \in [0,1]$ and

$i \in \{1,2\}$. Set, for abbreviation, $\overline{u} = \int_\Omega x_i u \ \mathrm{d}\mu$ and similarly for $\overline{v}$, then:

$$\begin{aligned}
\left(\int_\Omega x_i(tu + (1-t)v) \ \mathrm{d}\mu \right)^2 &= \left(t \int_\Omega x_i u \ \mathrm{d}\mu + (1-t) \int_\Omega x_i v \ \mathrm{d}\mu \right)^2 \\
&= t^2 \overline{u}^2 + 2t(1-t)\overline{uv} + (1-t)^2 \overline{v}^2 \\
&\leq t^2 \overline{u}^2 + t(1-t) \left[\overline{u}^2 + \overline{v}^2 \right] + (1-t)^2 \overline{v}^2 \\
&= \left[t^2 + t(1-t) \right] \overline{u}^2 + \left[(1-t)^2 + t(1-t) \right] \overline{v}^2 \\
&= t \left(\int_\Omega x_i u \ \mathrm{d}\mu \right)^2 + (1-t) \left(\int_\Omega x_i v \ \mathrm{d}\mu \right)^2 .
\end{aligned}$$

Suppose $f_0 \geq 0$, where $\mu(\{x \in \Omega : f(x) > 0\}) < \frac{1}{4}\mu(\Omega)$. Denote by $\mathscr{F}$ the set of rearrangements of f_0 on Ω. If $\alpha > 0$ is sufficiently large, then there are four distinct local maximisers of Ψ_α with respect to $\mathscr{F}$ concentrated in each quadrant of the unit disc, called $v_1, v_2, v_3, v_4 \in \mathscr{F}$. Now, suppose R_θ represents a rotation around the centre of θ radians, then set

$$v_{i+1} = v_i \circ R_{\frac{\pi}{2}} \text{ for all } i \in \{1,2,3\}.$$

Suppose $x_1 = (1,0), x_2 = (0,1), x_3 = (-1,0)$ and $x_4 = (0,-1) \in \partial B$.
It must be shown that

$$\begin{aligned}
\mathscr{C} &= \{h \in C(B; \mathscr{F}) : h(x_i) = v_i \text{ for all } i \in \{1, \ldots, 4\}\} \\
c &= \sup_{h \in \mathscr{C}} \inf_{y \in B} \Psi_\alpha(h(y))
\end{aligned}$$

satisfy

$$\inf_{\zeta \in L^p(\Omega)} \Psi_\alpha(\zeta) < c < \min \{\Psi_\alpha(v_1), \ldots, \Psi_\alpha(v_4)\}.$$

The left-hand infimum is, of course, zero. The second inequality must be proven.
Assume, seeking contradiction, that

$$c = \min \{\Psi_\alpha(v_1), \ldots, \Psi_\alpha(v_4)\}.$$

Since $v_1, \ldots, v_4 \in \mathscr{F}$ are distinct local maximisers of Ψ_α relative to $\mathscr{F}$, it must be that any path linking the rearrangement in $\{v_1, \ldots, v_4\}$ which has the minimum value of $\{\Psi_\alpha(v_1), \ldots, \Psi_\alpha(v_4)\}$, to any other element of this set cannot drop below the value of $\min \{\Psi_\alpha(v_1), \ldots, \Psi_\alpha(v_4)\}$. This contradicts the supposition that these rearrangements are local maximisers of Ψ_α relative to $\mathscr{F}$. By an application of Theorem 3.22, there must be a critical value of Ψ_α that is distinct from $v_1, \ldots, v_4$.

Chapter 4

Degree Theory

4.1 Topological Degree: An Axiomatic Treatment

This section studies basic problems concerning equations of the form $T(x) = y$, where $(Y, \|\cdot\|)$ is a Banach space, $T : Y \to Y$ is a compact map, and $y \in Y$ is a given point. The aim is to construct a theory that will provide multiplicity results for the solutions $x \in Y$. Degree theory may be treated axiomatically.

There are other treatments of degree theory available: [33] looks at degree theory for continuous functions, finite-dimensional spaces, Sobolev functions, and finally in infinite-dimensional spaces; whilst [23] spends the opening two chapters dealing with topological degree theory in finite and then infinite dimensions.

However, I will focus on N. G. Lloyd's perennial *Degree Theory* [53]. Another great text on degree theory and related problems is [56]. I define the concept of an admissible class of functions.

Definition 4.1 (Admissible Class)**.** Let Y be a normed space, and let $\mathscr{B}$ be a collection of open and bounded subsets of Y. To each $D \in \mathscr{B}$, assign a subset $M(D)$ of $C(\overline{D}; Y)$, the linear space of continuous mappings from $\overline{D}$ into Y, such that

(i) the identity mapping $\mathrm{id}_{\overline{D}} : \overline{D} \to \overline{D}$, $\mathrm{id}_{\overline{D}}(x) = x$ is a member of $M(D)$ for all $D \in \mathscr{B}$ where $D \neq \emptyset$,

(ii) if $D \subseteq D'$ and T is a member of $M(D')$, then $T|_{\overline{D}}$, the restriction of T to $\overline{D}$ is a member of $M(D)$,

(iii) if T is a member of $M(D)$, then $T - y$ is also a member of $M(D)$ for every $y \in Y$, where $T - y : \overline{D} \to Y$, $(T - y)(x) = T(x) - y$.

The *admissible class* of functions on Y is denoted $M(\mathscr{B}) = \{M(D) : D \in \mathscr{B}\}$.

A topological degree is the correspondence between a triple of a function, an open and bounded subset of the normed space Y, and a point in Y, and the integers, satisfying four axioms.

Definition 4.2 (Topological Degree)**.** Let Y be a normed space, and suppose $\mathscr{B}$ is a given collection of open and bounded subsets of Y, and $M(\mathscr{B})$ is an admissible class of functions. The function

$$d : \left\{(T, D, y) : T \in M(D),\ D \in \mathscr{B},\ y \in Y \setminus \overline{T(\partial D)}\right\} \to \mathbb{Z}$$

is called a *topological degree* for $M(\mathscr{B})$ if the following axioms are met:

(I) (Normalisation) If $D \in \mathscr{B}$, where $D \neq \emptyset$ and $y \in D$, then $d(\mathrm{id}, D, y) = 1$,

(II) (Additivity) If $D_1, D_2 \in \mathscr{B}$ are two disjoint subsets of D. If $T \in M(D)$ and the point satisfies $y \notin \overline{T(\overline{D} \setminus (D_1 \cup D_2))}$, then

$$d(T, D, y) = d(T, D_1, y) + d(T, D_2, y).$$

(III) (Homotopy Invariance) If $D \in \mathscr{B}$, where $D \neq \emptyset$, and $H(t)$ is a family of mappings in $M(D)$, which depend continuously on $t \in [0, 1]$. If

$$y \notin \overline{H(t)(\partial D)} \text{ for all } 0 \leq t \leq 1,$$

then $d(H(t), D, y)$ is independent of $t \in [0, 1]$.

(IV) (Translation Invariance) If $D \in \mathscr{B}$ and $D \neq \emptyset$, and $T \in M(D)$ with $y \in Y \setminus \overline{T(\partial D)}$, then

$$d(T, D, y) = d(T - y, D, 0).$$

In this case, $d(T, D, y)$ is described as the degree of T at y relative to D.

Remark 4.3. (i) In order to provide simplicity in the above definition, id is understood to be $\mathrm{id}|_{\overline{D}}$ in (I) and T refers to the functions $T|_{\overline{D}}$, $T|_{\overline{D_1}}$ and $T|_{\overline{D_2}}$. The context of usage will provide clarity as to which functions are meant.

(ii) Topological degrees do not necessarily exist for each given admissible class of functions. When they do exist, topological degrees are unique, as shown in [23]. However, I do not explicitly use the uniquness of this degree, and I am more interested in its existence and properties.

(iii) The axioms of Homotopy Invariance and Translation Invariance may be combined into a single axiom, as given in [52, Theorem 5.1.2 (pg. 74)]:
If $D \in \mathscr{B}$ and $D \neq \emptyset$, and $H : [0,1] \to M(D),\ y : [0,1] \to Y$ are both continuous functions, and if

$$y(t) \notin \overline{H(t)(\partial D)} \text{ for all } 0 \leq t \leq 1,$$

then $d(H(t), D, y(t))$ is independent of $t \in [0,1]$.

The axiom of Homotopy Invariance, Definition 4.2 (III), is very powerful. Properties of a topological degree that arise immediately from these axioms are detailed below.

Theorem 4.4. *Suppose $(Y, \|\cdot\|)$ is a normed space, and $\mathscr{B}$ is a collection of open and bounded subsets of Y, and $M(\mathscr{B})$ is an admissible class of functions on Y. Suppose further that d is a topological degree for $M(\mathscr{B})$. Let $D \in \mathscr{B}$ with $D \neq \emptyset$, $T \in M(D)$ and $y \in Y$ then*

(i) for all functions $T \in M(D)$ and points $y \in Y$, $d(T, \emptyset, y) = 0$;

(ii) if $U \subset D$ is an open subset of D, and $y \notin T(\overline{D} \setminus U)$, then

$$d(T, D, y) = d(T, U, y);$$

(iii) if $y \notin T(\overline{D})$, then $d(T, D, y) = 0$;

(iv) if $d(T, D, y) \neq 0$ and $T(\overline{D})$ is a closed set, then there exists $x \in D$ such that $T(x) = y$;

(v) if D is the disjoint union of open sets $D_1, \ldots, D_n$ for some $n \in \mathbb{N}$, and $y \notin \overline{T\left(\overline{D} \setminus \bigcup_{i=1}^n D_i\right)}$, then

$$d(T, D, y) = \sum_{i=1}^{n} d(T, D_i, y);$$

(vi) for all $q \in Y$, then

$$d(T, D, y) = d(T - q, D, y - q);$$

(vii) $d(T, D, \cdot)$ is constant on the path-connected components of $Y \setminus \overline{T(\partial D)}$;

(viii) if $M(D)$ is convex, then for each $T \in M(D)$, there exists $\varepsilon > 0$ such that if $S \in M(D)$ satisfies $\|T(x) - S(x)\| < \varepsilon$ for all $x \in D$, then

$$d(T, D, y) = d(S, D, y);$$

(ix) if $M(D)$ is convex, and if $S, T \in M(D)$ satisfies $S|_{\partial D} = T|_{\partial D}$, then

$$d(T, D, y) = d(S, D, y).$$

Proof. (i) Set $D_1 = D$ and $D_2 = \emptyset$ in Definition 4.2 (II), so $d(T, D, y) = d(T, D, y) + d(T, \emptyset, y)$, which means $d(T, \emptyset, y) = 0$.

(ii) Set $D_1 = U$ and $D_2 = \emptyset$, then

$$\begin{aligned} d(T, D, y) &= d(T, U, y) + d(T, \emptyset, y) && \text{by Definition 4.2 (II)} \\ &= d(T, U, y) + 0 && \text{by } (i) \\ &= d(T, U, y). \end{aligned}$$

(iii) Apply Definition 4.2 (II) with $D_1 = D_2 = \emptyset$, noting that $\overline{T(\overline{D})} = \overline{T(D)}$, so by (i):

$$d(T, D, y) = d(T, \emptyset, y) + d(T, \emptyset, y) = 0 + 0 = 0.$$

(iv) If there exists no $x \in D$ such that $T(x) = y$, then $y \notin T(D)$. By supposition, $y \notin \overline{T(\partial D)}$. By the closedness of $T(\overline{D})$, it follows that $y \notin T(D) \cup \overline{T(\partial D)} = \overline{T(D)}$. By (iii), $d(T, D, y) = 0$. The contrapositive of this statement is (iv).

(v) This follows by induction on Definition 4.2 (II).

(vi) Let $z \in Y$ be in the same path-connected component of $Y \setminus \overline{T(\partial D)}$, and let $\gamma : [0, 1] \to Y \setminus \overline{T(\partial D)}$ be a path between y and some $z \in Y$. Set

$$H : [0, 1] \to M(D), \;\; H(t) = T - \gamma(t),$$

which is a continuous mapping into $M(D)$. Notice that, for all $t \in [0, 1]$, $0 \notin \overline{H(t)(\partial D)}$.

$$\begin{aligned} d(T, D, y) &= d(T - y, D, 0) && \text{by Definition 4.2 (IV)} \\ &= d(H(0), D, 0) && \text{by construction of } H \\ &= d(H(1), D, 0) && \text{by Definition 4.2 (III)} \\ &= d(T - z, D, 0) && \text{by construction of } H \\ &= d(T, D, z) && \text{by Definition 4.2 (IV).} \end{aligned}$$

Hence,

$$\mathfrak{d} : Y \setminus \overline{\partial D} \to \mathbb{Z}, \;\; \mathfrak{d}(y) = d(T, D, y)$$

is a continuous map into the integers, making it constant on path-connected components of $Y \setminus \overline{T(\partial D)}$.

(vii) Since $y \notin \overline{T(\partial D)}$, I can set $\varepsilon = \text{dist}(y, T(\partial D)) > 0$. Let

$$H : [0,1] \to M(D),\ H(t) = t \cdot S + (1-t) \cdot T.$$

The convexity of $M(D)$ ensures that this map is well-defined. If $S \in M(D)$ satisfies $||S(x) - T(x)|| < \varepsilon$ for all $x \in D$, then for any $t \in [0,1]$, it follows

$$||H(t)(x) - T(x)|| = ||t \cdot (S-T)(x)|| < t \cdot \varepsilon \leq \varepsilon.$$

Thus, $y \notin \overline{H(t)(\partial D)}$, so by Definition 4.2 (III):

$$d(T, D, y) = d(H(0), D, y) = d(H(1), D, y) = d(S, D, y).$$

(viii) If $H : [0,1] \to M(D)$ is the straight line between S and T, then for all $t \in [0,1]$,

$$H(t)|_{\partial D} = tS|_{\partial D} + (1-t)T|_{\partial D} = tT|_{\partial D} + (1-t)T|_{\partial D} = T|_{\partial D},$$

so $y \notin \overline{H(t)(\partial D)} = \overline{T(\partial D)}$. By Definition 4.2 (III), $d(T, D, y) = d(S, D, y)$.

□

Example 4.5. The computation of the topological degree in one dimension is particularly simple. Let $f : (a,b) \to \mathbb{R}$ be continuous, then

$$d(f, (a,b), 0) = \begin{cases} 1 & \text{if } f(a) < 0 < f(b), \\ 0 & \text{if } f(a)f(b) > 0, \\ -1 & \text{if } f(a) > 0 > f(b). \end{cases}$$

It is through a consideration of these properties that the technical power of degree theory is revealed. Our initial problem in this section was finding solutions to the problem $T(x) = y$, which permeates through the studies of Functional and Real Analysis. In particular, Theorem 4.4 (*iv*) means that if it is discovered for some open and bounded subset D of Y that $d(T, D, y) \neq 0$, where $T(\overline{D})$ is a closed set in Y, then there exists a solution to the problem; that is, there exists $x \in D$ such that $T(x) = y$. There are other methods for discovering solutions, such as variational methods, as examined in [69]. These methods are potent, but often arduous and unyielding. Thanks to results such as Theorem 4.4(v), degree theory reduces the multiplicity problem to a simple integer counting game. It is this power that I wish to harness, but I require the ability to quickly calculate the degree of functions at a given point relative to given open sets. The usage of the degree theory in the study of fixed points will be shown later in this section. The actual construction of degrees of compact perturbations from the identity, called the *Leray-Schauder Degree*, requires the use of a finite-dimensional degree: the *Brouwer Degree*.

It is for this reason that I will not include the full construction of the Leray-Schauder degree here. The Brouwer degree is built by the following series of admitted functions and points:

(i) $f \in C^1(\overline{\Omega}; \mathbb{R}^N)$ and y is a regular value of f and $y \notin f(\partial\Omega)$;

(ii) $f \in C^2(\overline{\Omega}; \mathbb{R}^N)$ and any $y \notin f(\partial\Omega)$ is allowed;

(iii) $f \in C(\overline{\Omega}; \mathbb{R}^N)$ and any $y \notin f(\partial\Omega)$ is allowed.

The first chapter of [23], the second chapter of [52] and [33] all consider the construction of the Brouwer degree. This degree may be extended to infinite-dimensional normed spaces, or it may be extended to continuous functions between connected, orientated, smooth N-manifolds, as in [**?** 60]. There are also a variety of texts on infinite-dimensional Morse Theory, such as [17].

4.2 Leray-Schauder Degree: Infinite-Dimensional Case

Before I begin, I must show that the admissible class of continuous functions between an infinite dimensional normed space and itself does *not* permit a topological degree. I will use the example given in [52].

Let Y be the convex subset of the Banach space of continuous functions, where the functions have the domain $[0, 1]$ and the range $[0, 1]$, with the supremum norm $y : [0, 1] \to [0, 1]$, $\|y\| = \max_{s \in [0,1]} |y(s)|$. Let y_0 be the constant function $y_0(s) = \frac{1}{2}$ for all $s \in [0, 1]$, and let $D = \left\{y \in Y : \|y - y_0\| < \frac{1}{2}\right\}$. Note that D is an open and bounded subset of Y. Choose $\phi \in Y$ such that $0 \leq \phi(s) \leq 1$ for all $s \in [0, 1]$, $\phi(0) = 0$ and $\phi(1) = 1$. Define

$$\Phi : \overline{D} \to Y, \ \Phi(y) = \phi \circ y.$$

Hence, $\Phi(\overline{D}) \subseteq \overline{D}$. Consider the homotopy

$$H : [0, 1] \times \overline{D} \to Y, \ H(t, y) = t\Phi(y) + (1 - t)y.$$

If $y \in \partial D$, then $\|y - y_0\| = \frac{1}{2}$, so $0 \leq y(s) \leq 1$, for all $s \in [0,1]$. Thus, there must be some $s_0 \in [0,1]$ such that $y(s_0) = 0$ or $y(s_0) = 1$. If $y(s_0) = 0$, then $H(t,y)(s_0) = 0$ for all $t \in [0,1]$. Similarly, if $y(s_1) = 1$, then $H(t,y)(s_1) = 1$ for all $t \in [0,1]$. Since $0 \leq \phi(s) \leq 1$ implies that $0 \leq H(t,y)(s) \leq 1$ for all $t, s \in [0,1]$, it follows that $y \in \partial D$ implies $H(t,y) \in \partial D$ for all $t \in [0,1]$. Assume, seeking contradiction, that a topological degree d may be defined on Y with the set of continuous functions $[0,1] \to [0,1]$ being the admissible class of functions. Let $z \in D$, and since $H(t, \partial D) \subseteq \partial D$ for all $t \in [0,1]$, homotopy invariance implies that $d(\Phi, D, z) = d(\mathrm{id}_D, D, z) = 1$. By the validity of the degree, there must exist a solution $x \in D$ such that $\Phi(x) = z$. This assumption is flawed. Set $z : [0,1] \to [0,1]$, $z(s) = \frac{1}{4} + \frac{1}{2}s$, and $\phi : [0,1] \to [0,1]$ by

$$\phi(s) = \begin{cases} s, & \text{for } 0 \leq s \leq \frac{1}{2} \\ 1 - s, & \text{for } \frac{1}{2} \leq s \leq \frac{5}{8} \\ \frac{5}{3}(s-1) + 1, & \text{for } \frac{5}{8} \leq s \leq 1. \end{cases}$$

If $x \in D$ is a solution to $\Phi(x) = z$, so $\phi(x(0)) = \frac{1}{4}$, whence $x(0) = \frac{1}{4}$. $z(s)$ increases from $\frac{1}{4}$ to $\frac{3}{4}$, but $\phi(x(s))$ can increase from $\frac{1}{4}$ by at most $\frac{1}{4}$ before the function starts to decrease. Thus, no such $x \in D$ exists. Hence, a topological degree cannot be defined on Y for the admissible class of continuous functions.

A major difference between finite and infinite dimensional normed spaces is that bounded sets in infinite dimensional normed spaces are not necessarily precompact. It is because of this observation that the admissible class of continuous functions on infinite dimensional normed spaces fail to permit a topological degree. Compact maps into locally compact Haussdorff spaces are necessarily closed maps, by [61, Lemma 1]. This insight simplifies the considerations around a topological degree theory for compact perturbations.

I will provide a definition of the Leray-Schauder degree, and state that it is a topological degree on the admissible class of compact perturbations from the identity.

Definition 4.6 (Leray-Schauder Degree)**.** Suppose $(X, \|\cdot\|)$ is a Banach space, and Ω is an open and bounded subset of X. Suppose $f : \overline{\Omega} \to X, f = \mathrm{id}_X - T$, where $T : \overline{\Omega} \to X$ is compact. Suppose $\hat{T} : \overline{\Omega} \to X$ is a continuous map with finite dimensional range, such that

$$\left\| T(x) - \hat{T}(x) \right\| < \mathrm{dist}(x, f(\partial\Omega)) \text{ for all } x \in \overline{\Omega}.$$

Choose a finite dimensional linear space V which contains $\hat{T}(\overline{\Omega})$ and $y \in X \setminus f(\partial\Omega)$, let $\Omega_V = \Omega \cap V$, and set $\hat{f} : \overline{\Omega_V} \to V$, $\hat{f}(x) = x - \hat{T}(x)$. The *Leray-Schauder degree* is defined as

$$d(f, \Omega, y) = d(\hat{f}, \Omega_V, y),$$

where $d(\hat{f}, \Omega_V, y)$ is the Brouwer degree.

Remark 4.7. It is shown in [52, Lemma 4.2.4] that the value of $d(\hat{f}, \Omega_V, y)$ is indepedent of the finite-dimensional subspace V used.

Theorem 4.8. *Suppose X is a Banach space, then the Leray-Schauder Degree, d in Definition 4.6, is a topological degree for the admissible class of functions $\mathscr{K}_1(\Omega)$, where Ω is an open and bounded subset of X.*

Proof. The existence and uniqueness of the Leray-Schauder degree has been proved in many places, such as [52, Chapter 4], [33, Chapter 7] and [23, Chapter 8]. $\square$

Degree theory can then be used to prove results involving fixed points.

4.3 Fixed Point Theorems

From a consideration of problems $f : Y \to Y$ and $y = f(x)$, it is also pertinent to find points in Y such that $f(y) = y$. These points are called fixed points.

Definition 4.9 (Fixed Point)**.** Let Y be a set, and suppose $f : Y \to Y$ is a function. A point $y \in Y$ is called a *fixed point of f* if $f(y) = y$. The set of fixed points of f is denoted $\mathrm{Fix} f \subset Y$.

Remark 4.10. The fixed point of a compact map is the zero of a compact perturbation from the identity, as for any function $f : Y \to Y$ for a vector space Y, it follows

$$\mathrm{Fix} f = \{y \in Y : f(y) = y\} = \{y \in Y : y - f(y) = 0\} = (\mathrm{id}_Y - f)^{-1}(0).$$

Many authors have proved fixed point theorems, which establish the existence of fixed points under certain suppositions, and degree theory is a powerful tool for proving when certain maps have fixed points.

Brouwer's Fixed Point Theorem may be generalised to infinite-dimensional Banach spaces, and it is usually called Schauder's Fixed Point Theorem. This theorem will be invaluable in solving the boundary value problems discussed in the next section.

Lemma 4.11. *Let $(Y, \|\cdot\|_Y)$ be a real Banach space, and let $\Omega \subset Y$ be a bounded and open subset with $0 \in \Omega$. Suppose $f : \overline{\Omega} \to Y$ is a compact map satisfying*

$$f(y) \neq \lambda y, \text{ for all } y \in \partial\Omega \text{ and } \lambda \geq 1.$$

Then f has a fixed point.

Proof. Consider the homotopy:

$$H : \overline{\Omega} \times [0, 1] \to Y, \ H(y, t) = y - tf(y).$$

This is a homotopy of compact transformations perturbing the identity. Assume, seeking contradition, that $0 \in H(\partial\Omega \times [0, 1])$, so there exists $t \in [0, 1]$ and $x \in \partial\Omega$ such that $0 = x - tf(x)$, that is, $tf(x) = x$. If $t = 0$, then $0 \in \partial\Omega$, which is a contradiction as Ω is open and $0 \in \Omega$. If $0 < t \leq 1$, then $f(x) = t^{-1}x$, which contradicts the supposition, as $t^{-1} \geq 1$. Hence, $0 \in H(\partial\Omega \times [0, 1])$. It follows that, for the Leray-Schauder degree:

$$\begin{aligned} d(\mathrm{id} - f, \Omega, 0) &= d(\mathrm{id}, \Omega, 0) && \text{by Definition 4.2 (III)} \\ &= 1 && \text{by Definition 4.2 (I), as } 0 \in \Omega. \end{aligned}$$

By Theorem 4.4 (iv), it follows that f has a fixed point in $\overline{\Omega}$. □

Theorem 4.12 (Schauder's Fixed Point Theorem)**.** *Let* $(Y, \|\cdot\|_Y)$ *be a real Banach space, and let* $C \subset Y$ *be a non-empty, closed, bounded and convex subset. Suppose* $f : C \to C$ *is a compact map, then* f *has a fixed point in* C*.*

Proof. Since C is closed, bounded and convex in the Banach space Y, [12, Theorem 1] implies there exists a retraction $R : Y \to C$, so $R(y) = y$ for all $y \in C$. Let $\alpha > 0$ be sufficiently large such that $C \subset \overline{\mathbb{B}(0, \alpha)}$ and $C \cap \partial\mathbb{B}(0, \alpha) = \emptyset$. Set the function

$$g : \overline{\mathbb{B}(0, \alpha)} \to \overline{\mathbb{B}(0, \alpha)}, \ g(y) = (f \circ R)(y),$$

which is compact, as a composition of a compact and a continuous map.
Now, $\mathbb{B}(0, \alpha)$ is an open and bounded subset of Y, which contains 0. If there exists $y \in \partial\mathbb{B}(0, \alpha)$ such that $g(y) = y$, there is nothing more to prove, so I assume otherwise. Assume, seeking contradiction, that $g(y) = \lambda y$ for some $y \in \partial\mathbb{B}(y, \alpha)$ and $\lambda > 1$. Applying norms yields

$$\|g(y)\|_Y = \lambda \|y\|_Y > \|y\|_Y = \rho,$$

which is a contradiction, since $g\left(\overline{\mathbb{B}(0, \alpha)}\right) \subseteq C \subseteq \overline{\mathbb{B}(0, \alpha)}$.
By an application of Lemma 4.11, there exists $y \in \overline{\mathbb{B}(0, \alpha)}$ such that $g(y) = y$. Since $y = g(y) \in C$, then $R(y) = y$. Finally, $f(y) = (f \circ R)(y) = g(y) = y$, so f must have a fixed point. □

There are variations around this result, which prove the existence of fixed points with differing criteria placed on the domains.

Theorem 4.13. *Let $(Y, \|\cdot\|_Y)$ be a real Banach space, and let $C \subset Y$ be a closed and bounded subset of Y with a non-empty interior. Suppose $f : C \to C$ is compact, and if there exists $w \in D := \operatorname{int} C$ such that*

$$f(y) - w \neq \lambda(y - w) \text{ for all } \lambda > 1 \text{ and } y \in \partial D.$$

Then f has a fixed point.

Proof. I follow [52, Theorem 4.4.3]. Consider the following homotopy:

$$H : \overline{D} \times [0, 1] \to Y,\ H(y, t) = y - w - t(f(y) - w).$$

It is natural to exclude the possibility of a fixed point of f in ∂D; otherwise, the work would be done.
If $0 \in H(\partial D \times [0, 1))$, then $t(f(x) - w)) = x - w$ for some $t \in [0, 1)$ and $x \in \partial D$. Now, $t = 0$ would imply that $x = w$, but $x \in \partial D$ and $w \in D$, which is absurd, as $D \cap \partial D = \emptyset$. If $0 < t < 1$, then $f(x) - w = t^{-1}(x - w)$, contravening the suppositions on f.
Therefore,

$$\begin{aligned} d(\mathrm{id} - f, D, 0) &= d(\mathrm{id} - w, D, 0) && \text{by Definition 4.2 (III)} \\ &= d(\mathrm{id}, D, w) && \text{by Definition 4.2 (IV)} \\ &= 1 && \text{by Definition 4.2 (I), as } w \in D. \end{aligned}$$

By the validity of the Leray-Schauder degree, Theorem 4.4 (*iv*) means that there exists $y \in \overline{D}$ such that $f(y) = y$. ☐

Definition 4.14 (Isolated, Local Degree)**.** Let $(Y, \|\cdot\|)$ be a real Banach space, and let $\Omega \subset Y$ be a closed and bounded subset of Y. Suppose $f : \overline{\Omega} \to Y$ is compact. A fixed point $x \in \Omega$ of f is *isolated* if there exists an open set $U_x \subset \Omega$ such that

$$\left(\overline{U_x} \setminus \{x\}\right) \cap \operatorname{Fix} f = \emptyset.$$

The *local degree* of an isolated fixed point, or *index*, of $x \in \Omega$ is equal to $d(\mathrm{id} - f, U_x, 0)$, where U_x is an open subset of Ω such that $\left(\overline{U_x} \setminus \{x\}\right) \cap \operatorname{Fix} f = \emptyset$. This local degree is denoted $\operatorname{ind}(f, x)$.

Remark 4.15. It is necessary to demonstrate that the definition of the local degree is well-defined. Suppose $U_x \subset \Omega$ is an open subset that contains a fixed point of f called $x \in \Omega$, and U'_x is an open subset of U_x that contains x. By definition, $0 \notin (\mathrm{id} - f)(\overline{U_x})$.

Note that $U_x \setminus \overline{U'_x}$ does not contain x, so by the properties of the Leray-Schauder degree:

$$\begin{aligned} d(\mathrm{id} - f, U_x, 0) &= d(\mathrm{id} - f, U'_x, 0) + d(\mathrm{id} - f, U_x \setminus \overline{U'_x}, 0) && \text{by (II) of Definition 4.2} \\ &= d(\mathrm{id} - f, U'_x, 0) && \text{by (I) of Definition 4.2.} \end{aligned}$$

This means all open subsets that contain one fixed point but no others have the same degree, making the definition well-defined.

For a compact map, the finiteness of the multiplicity of fixed points is equivalent to all the fixed points being isolated.

Proposition 4.16. *Suppose $(Y, \|\cdot\|)$ is a real Banach space and $\Omega \subset Y$ is a nonempty, open and bounded subset of Y, and let $f : \overline{\Omega} \to Y$ be a compact map. Then f has a finite number of fixed points if, and only if, all of the fixed points of f are isolated.*

Proof. (Necessity) Suppose that f has a finite number of fixed points, labelled $x_1, \ldots, x_n$ for some $n \in \mathbb{N}$. Each fixed point may be placed inside an open set, and these open sets contains no other fixed points. For each $i \in \{1, \ldots, n\}$,

$$U_i = \mathbb{B}(x_i, \alpha) \subset Y, \text{ where } r = \frac{2}{5} \min \left\{ \left\| x_j - x_{j'} \right\| : j, j' \in \{1, \ldots, n\} \text{ and } j \neq j' \right\}.$$

By construction, each U_i is open and only contains a single fixed point of f. Hence, all of the fixed points of f are isolated.

(Sufficiency) Suppose all of the fixed points of f are isolated. Assume, seeking contradiction, that f has an infinite number of points. Enumerate some of these fixed points as $(x_n)_{n\in\mathbb{N}} \subset \Omega$. By sequential compactness, there exists a convergent subsequence $(x_{n_j})_{j\in\mathbb{N}}$ with limit $x \in Y$, that is, $\lim_{j\to\infty} x_{n_j} = x$. By continuity of f, it holds that:

$$f(x) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} x_{n_j} = x.$$

Hence, this x is a fixed point of f. However, x cannot be isolated, by the definition of limits. This is a contradiction. Thus, f must have a finite number of fixed points. □

Finally, it is noted that the minimum number of fixed points has a simple lower bound.

Lemma 4.17. *Suppose $(Y, \|\cdot\|)$ is a real Banach space, and $\Omega \subset Y$ is a nonempty, bounded and open subset of Y. Suppose $f : \overline{\Omega} \to Y$ is compact and has finitely many fixed points. Then*

$$|\{x \in \mathrm{Fix}\ f : \mathrm{ind}(f, x) = 1\}| = d(id - f, \Omega, 0) + |\{x \in \mathrm{Fix}\ f : \mathrm{ind}(f, x) = -1\},$$

where $\operatorname{ind}(f,x)$ *is the local degree of the fixed point* $x \in \Omega$*.*

Proof. Note that $\operatorname{ind}(f,x)$ is well-defined, since there are finitely many fixed points, so all of them are isolated. Suppose the fixed points may be isolated inside open sets called U_x for each $x \in \Omega$. Recall from the calculations of the Brouwer degree that the index must be equal to either 1, 0 or -1. It follows

$$\begin{aligned} d(\mathrm{id}-f,\Omega,0) &= \sum_{x\in \mathrm{Fix}\ f} d(\mathrm{id}-f,U_x,0) \\ &= |\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=1\}\,| + (-1)|\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=-1\}\,| + 0 \\ &= |\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=1\}\,| - |\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=-1\}\,|. \end{aligned}$$

Consequently,

$$|\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=1\}\,| = d(\mathrm{id}-f,\Omega,0) + |\,\{x \in \mathrm{Fix}\ f : \operatorname{ind}(f,x)=-1\}\,.$$

$\square$

Chapter 5

The Boundary Value Problem

5.1 Local Maximisers of the Energy Functional

Suppose $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, for some $N \in \mathbb{N}$, $\mathscr{F}$ is the set of rearrangements on Ω of a prescribed function $f_0 \in L^p(\Omega)$. We are concerned with the existence and multiplicity of solutions for boundary value problems of the form:

$$\left.\begin{aligned} -\Delta\psi &= \varphi \circ \psi, \\ \psi &\in H_0^1(\Omega), \\ -\Delta\psi &\in \mathscr{F} \end{aligned}\right\}. \tag{5.1}$$

Here, the function, often called the *vorticity function*, $\varphi : \mathbb{R} \to \mathbb{R}$ is unknown *a priori*, but for the solutions to these problems, it will be discovered that φ is increasing, that is, non-decreasing.

The effect of the domain shape on the number of solutions has studied by E. N. Dancer [19, 20]. In those papers, the function φ is labelled f, it is multiplied by a fixed λ and the function is already known. Particular attention is paid to be the cases where $\varphi(y) = \exp(y)$ or $f(y) = y^\alpha$ for $1 < \alpha < \frac{N+2}{N-2}$, where $\Omega \subseteq \mathbb{R}^N$ for some $N > 1$.

This thesis will examine the number of solutions where the domain is approximately the union of a given number of balls. This adds to the growing corpus of work using topological degree theory to find solutions to boundary value problems, such as in [6, 48, 54, 68, 72]. Arnol′d proposed the variational principle for stable steady vortices on “isovortical surfaces” – a set of flows whose vortices are rearrangements of one another – in [7].

For clarity, it should be explained what the Sobolev spaces discussed in this chapter are. These definitions are provided by [2].

Definition 5.1 (Sobolev Norm, Sobolev Spaces)**.** Given $m \in \mathbb{N}$, a domain Ω and $1 \leq p < \infty$, the *Sobolev norms* are defined as:

$$\|u\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty,$$
$$\|u\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_\infty.$$

for any function u where the right-hand side makes sense. As normal, the $\|\cdot\|_p$ refers to the norm of $L^p(\Omega)$.

The following three spaces are called *Sobolev spaces*:

(a) $H^{m,p}(\Omega)$ is the completion of $\left\{ u \in C^m(\Omega) : \|u\|_{m,p} < \infty \right\}$ with respect to the norm $\|\cdot\|_{m,p}$.

(b) $W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\}$, where $D^\alpha u$ is the weak (or distributional) partial derivative, as in [2, Paragraph 1.62].

(c) $W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{m,p}(\Omega)$.

Now, I will define the energy functional.

Definition 5.2 (Energy Functional)**.** Let $\Omega \subset \mathbb{R}^N$ be a non-empty open and bounded set, where $\partial\Omega$ is a $C^{1,1}$ boundary. Suppose $N \geq 2$ and $\frac{2N}{N+2} \leq p < \infty$.

Let $K : L^p(\Omega) \to H_0^1(\Omega)$ be the linear inverse of the weak negative Laplacian. This is defined as, for all $\zeta \in L^p(\Omega)$,

$$\int_\Omega K\zeta \left(-\Delta\xi\right) \, \mathrm{d}\mu = \int_\Omega \zeta\xi \, \mathrm{d}\mu \text{ for all } \xi \in C_c^\infty(\Omega);$$
$$K\zeta \in H_0^1(\Omega).$$

Then the *energy functional* is defined to be

$$\mathscr{E} : L^p(\Omega) \to \mathbb{R}, \; \mathscr{E}(\zeta) = \frac{1}{2} \int_\Omega \zeta K\zeta \, \mathrm{d}\mu.$$

Remark 5.3. This definition may be generalised, as in [16], and so can correspond to different physical problems. Let $1 \leq p < \infty$, let q be the conjugate exponent of p, and denote by

$$\mathscr{L}u(x) = \sum_{1 \leq |\alpha| \leq M} a^\alpha(x) D^\alpha u(x),$$

a linear partial differential operator on Ω. Each a^α is measurable function, where $1 \leq |\alpha| \leq M$, so there is no 0-th order term. Let $K : L^p(\Omega) \to L^q(\Omega)$ be a compact, symmetric and linear operator. Denote

$$\langle v, w\rangle = \int_\Omega vw \; \mathrm{d}\mu \text{ for all } v \in L^p(\Omega),\; w \in L^q(\Omega).$$

A *symmetric* operator satisfies

$$\langle v, Kw\rangle = \langle w, Kv\rangle \text{ for all } v, w \in L^p(\Omega),$$

where μ is a finite positive measure on Ω equivalent to the N-dimensional Lebesgue measure. Also, for a given $m \in \mathbb{N}$, K satisfies $Kv \in W^m(\Omega)$ and $\mathscr{L}Kv = v$ almost everywhere in Ω for all $v \in L^p(\Omega)$. Let $w \in L^q(\Omega) \cap W^m(\Omega)$ satisfy $\mathscr{L}w = 0$ almost everywhere in Ω. The energy functional is then

$$\mathscr{E} : L^p(\Omega) \to \mathbb{R},\; \mathscr{E}(\zeta) = \frac{1}{2}\int_\Omega \zeta K\zeta \; \mathrm{d}\mu + \int_\Omega \zeta w \; \mathrm{d}\mu.$$

In Definition 5.2, the restrictions on p, related to N, are made to ensure $K\zeta \in W^{2,p}(\Omega)$ for all $\zeta \in L^p(\Omega)$.

Similar problems concerning the energy functional have been studied in [15, 16]. The non-linear term $\varphi : \mathbb{R} \to \mathbb{R}$ from equation (5.1) will be obtained from a linear maximisation problem, which is discussed below. The following result is an amalgamation of results from [15, 16].

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty open and bounded set, with the requirements on p and N as in Definition 5.2. Let $f_0 \in L^p(\Omega)$, and let $\mathscr{F}$ denote the set of rearrangements of f_0 on Ω. The following statements are equivalent for $v \in \mathscr{F}$:*

(i) $\mathscr{E}(w) \le \mathscr{E}(v)$ for all $w \in U$, where U is a strong neighbourhood of v relative to $\mathscr{F}$,

(ii) v is the unique maximiser of the linear functional

$$\Psi_v : \overline{\mathscr{F}^w} \to \mathbb{R},\; \Psi_v(w) = \langle w, Kv\rangle\,,$$

(iii) There exists an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $v = \varphi \circ Kv$ almost everywhere in Ω.

Proof. The strategy for this proof is $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.
$\underline{(i) \Rightarrow (ii)}$ This implication is detailed in [16, Theorem 3.3 (iii)]. Note that Ω, as a non-empty, bounded and open subset of $\mathbb{R}^N$, is a measure interval, and the Gâteaux derivative of $\mathscr{E}$ is $\mathscr{E}'(w) = Kw$.
$\underline{(ii) \Rightarrow (iii)}$ If v maximises $\langle \cdot, Kv\rangle$ relative to $\mathscr{F}^w$, $Kv \in L^q(\Omega) \cap W^2(\Omega)$ and $-\Delta Kv =$

$v \geq v$, [16, Lemma 2.15] implies that $v \in \mathscr{F}$ and there exists increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $v = \varphi \circ Kv$ almost everywhere in Ω, as required.
(iii) ⇒ (ii) Given there exists an increasing functions φ such that $v = \varphi \circ Kv$, the result [15, Lemma 3] implies, for each $w \in \mathscr{F}$:

$$\int_\Omega vKv \ \mathrm{d}\mu = \int_0^{\mu(\Omega)} v^\triangle (Kv)^\triangle \ \mathrm{d}\mu_L \geq \int_\Omega wKv \ \mathrm{d}\mu.$$

Let $w \in \overline{\mathscr{F}^w}$, so there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $\mathscr{F}$ such that $w_n \rightharpoonup w$. Then

$$\begin{aligned} \int_\Omega vKv \ \mathrm{d}\mu &\geq \lim_{n\to\infty} \int_\Omega w_n Kv \ \mathrm{d}\mu && \text{by prior calculation} \\ &= \int_\Omega wKv \ \mathrm{d}\mu \end{aligned}$$

Thus, v maximises Ψ_v relative to $\overline{\mathscr{F}^w}$.
(ii) ⇒ (i) By the definition of Gateaux differentiability, there exists a neighbourhood U of v in $\mathscr{F}$, such that, for all $w \in U$:

$$\begin{aligned} \mathscr{E}(w) - \mathscr{E}(v) &\leq \langle w - v, Kv \rangle && \text{by definition of differentiability} \\ &\leq 0 && \text{as } v \text{ is the maximiser of } \Psi_v \text{ relative to } \mathscr{F} \end{aligned}$$

Hence, $\mathscr{E}(w) \leq \mathscr{E}(v)$ for all $w \in U$. □

These characterisations expresses the local maximiser of the energy functional as the global maximiser of a linear functional.

Lemma 5.5. *Let Y be a Banach space, and let $\Psi : Y \to \mathbb{R}$ be a linear functional, so that*

$$\Psi(\alpha v + \beta w) = \alpha\Psi(v) + \beta\Psi(w) \text{ for all } \alpha, \beta \in \mathbb{R} \text{ and } v, w \in Y.$$

Suppose C is a non-empty and convex subset of Y. If C contains a maximiser of Ψ, and the maximiser is either unique or there are uncountably many maxmisers.

Proof. If there exist two distinct maximisers $u \neq v$ and $u, v \in C$. Then define

$$\gamma : [0,1] \to C, \ \gamma(t) = tu + (1-t)v.$$

This is well-defined, as C is convex, and set $\Psi(u) = \Psi(v) = M$. Then for all $t \in [0,1]$:

$$\Psi(\gamma(t)) = t\Psi(u) + (1-t)\Psi(v) = tM + (1-t)M = M.$$

Hence, there are uncountably many distinct maximisers. □

Remark 5.6. The existence of a maximiser may be deduced by the use of Weierstrass's Theorem. This theorem requires compactness, so if Y is a reflexive Banach space, then the weak compactness of C is sufficient.

It is required, for the definition of the *maximiser function*, that the global maximiser of the linear function $\Psi_v : \overline{\mathscr{F}^w} \to \mathbb{R},\ \Psi_v(w) = \int_\Omega wKv \,\mathrm{d}\mu$ is unique.

Lemma 5.7. *Let Y be a normed space, and let $C \subset Y$ be a non-empty, closed and convex set. Let $\Psi : C \to \mathbb{R}$ be a convex function, that is:*

$$\Psi(tu + (1-t)v) \leq t\Psi(u) + (1-t)\Psi(v) \text{ for all } u, v \in C \text{ and } t \in (0,1).$$

Suppose $u^* \in C$ is a maximiser of Ψ. If Ψ is non-constant, then $u^* \in \partial C$.

Proof. Suppose Ψ is not constant, and assume, seeking contradiction, that $u^* \notin \partial C$. Then u^* lies in the interior of C, which is also convex, and since Ψ is non-constant, there exists $u \in C$ such that $\Psi(u) < \Psi(u^*)$. There also exists $v \in C$, for which there exists $\lambda \in (0,1)$, such that $u^* = \lambda u + (1-\lambda)v$. Ψ is convex, so

$$\Psi(u^*) \leq \lambda\Psi(u) + (1-\lambda)\Psi(v) < \lambda\Psi(u^*) + (1-\lambda)\Psi(u^*) = \Psi(u^*),$$

which is a contradiction. Hence, $u^* \in \partial C$. □

Lemma 5.8. *Let L be a 2nd order weak differential operator and let $\Omega \subset \mathbb{R}^N$ be a non-empty and bounded domain, with a $C^{1,1}$-domain, for some $N \geq 2,\ N \in \mathbb{N}$; that is,*

$$Lw(x) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}D_{ij}w(x) + \sum_{i=1}^{N} b_i D_i w(x) + c(x)w(x),$$

for some functions $a_{ij}, b_i \in C(\overline{\Omega})$ where $1 \leq i, j \leq N$ and $w \in C^2(\Omega)$.

Suppose $u \in W^{2,p}(\Omega) \cap C(\overline{\Omega})$ for some $p > \frac{N}{2}$,

$$Lu(x) > 0 \text{ for all } x \in \Omega \text{ and } c \equiv 0 \text{ in } \Omega,$$

then all of the level sets of u have zero Lebesgue measure.

Proof. Assume, seeking contradiction, that there exists $\alpha \in \mathbb{R}$ such that $\{y \in \Omega : u(y) = \alpha\}$ has non-zero Lebesgue measure. By [36, Lemma 7.7], the weak derivatives are $D_i u(y) = 0$ and $D_{ij}u(y) = 0$ for all $1 \leq i, j \leq N$, since when a function is strongly differentiable, their weak and strong derivatives agree. Hence, $Lu(y) = 0$. This is a contradiction. □

Before proceeding, I should justify the constraints on $p,\ N$ and Ω in this problem. Firstly, it is required that $1 < p < \infty$ and q satisfies $p^{-1} + q^{-1} = 1$. The Sobolev space

$H_0^1(\Omega)$ is needed to be imbedded into $L^q(\Omega)$. Since $H_0^1(\Omega) = W_0^{1,2}(\Omega)$, the Sobolev Embedding Theorem, such as [36, Corollary 7.11] implies that:

(i) if $N = 2$, $W_0^{1,2}(\Omega) \to L^q(\Omega)$ for any $1 < q < \infty$, and hence any $1 < p < \infty$,

(ii) if $N > 2$, $W_0^{1,2}(\Omega) \to L^q(\Omega)$ for any $q \leq 2^* = \frac{2N}{N-2}$. In this latter case, $1 < q \leq \frac{2N}{N-2}$ implies

$$
\begin{aligned}
1 - \frac{1}{p} &= \frac{1}{q} \\
&\geq \frac{N-2}{2N} \\
&= \frac{1}{2} - \frac{1}{N}
\end{aligned}
$$

so $1 < \frac{2N}{N+2} \leq p < \infty$.

Moreover, it is desired to express the supremum of each Kv, $v \in L^p(\Omega)$, in terms of the p-norm of Kv, so it is sufficient to ensure each Kv is continuous. There are several global regularity results for strong solutions, so applying the following lemma:

Lemma 5.9. *Let Ω be a bounded $C^{1,1}$-domain in $\mathbb{R}^N$, and let the operator L be strictly elliptic in Ω with coefficients $a_{ij} \in C(\overline{\Omega})$, $b_i, c \in L^\infty(\Omega)$, $i, j \in \{1, \ldots, N\}$ and $c \leq 0$. Then, if $f \in L^p(\Omega)$, $p > \frac{N}{2}$, $\varphi \in C(\partial\Omega)$, the Dirichlet problem:*

$$
\begin{cases} Lu = f \text{ in } \Omega, \\ \ \ u = \varphi \text{ on } \Omega \end{cases}
$$

has a unique solution $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$.

Proof. This result is [36, Corollary 9.18]. □

The requirement placed on p is that $p > \frac{N}{2}$, which is stronger than the other restrictions, that is,

$$
1 \leq \frac{2N}{N+2} \leq \frac{N}{2} \text{ for } N \in \mathbb{N} \text{ and } N \geq 2
$$

with equality only when $N = 2$. The requirements on Ω are that it has a $C^{1,1}$ boundary.

Now, I can describe the maximiser function in a well-defined way.

Definition 5.10 (Maximiser Function)**.** Let $\Omega \subset \mathbb{R}^N$ be a non-empty open and bounded set, where $N \geq 2$, $N \in \mathbb{N}$ and $\frac{2N}{N+2} < p < \infty$. Let $K : L^p(\Omega) \to H_0^1(\Omega)$ be the linear inverse of the negative Laplacian with zero Dirichlet boundary conditions. Also, let

$f_0 \in L^p(\Omega)$, and $f_0 > 0$ throughout Ω, and denote the set of rearrangements of f_0 on Ω by $\mathscr{F}$. The *maximiser function* is defined as

$$F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w},\ F(v) = \arg\max_{w \in \overline{\mathscr{F}^w}} \int_\Omega wKv \, \mathrm{d}\mu.$$

Proposition 5.11. *If the conditions of Definition 5.10 are met, then F is a well-defined and strongly continuous function. Also, the image of F is a compact subset of $\mathscr{F}$.*

Proof. Given $f_0 > 0$ throughout Ω, [15, Lemma 5] shows that $f > 0$ throughout Ω for all $f \in \overline{\mathscr{F}^w}$. Since $\Delta(Kf)(x) = -f(x) < 0$ for all $f \in \overline{\mathscr{F}^w}$ and $x \in \Omega$, Lemma 5.8 implies that, setting $L = \Delta$, all of the level sets of Kf have zero Lebesgue measure. Fix $f \in \mathscr{F}$. By [16, Lemma 2.9], there is an increasing function

$$\varphi : \mathbb{R} \to \mathbb{R},\ \varphi(s) = f^{\triangle}\left(\mu(\{x \in \Omega : Kf(x) \geq s\})\right)$$

such that $\varphi \circ Kf$ is a rearrangement of f, that is, $\varphi \circ Kf \in \mathscr{F}$. It follows, by [15, Theorem 3], that $f^* = \varphi \circ Kf$ is the unique maximiser of the functional $\langle \cdot\,, Kf\rangle$ over $\overline{\mathscr{F}^w}$. Hence, F is well-defined and $F(f) = f^*$.

Since each integral $\langle \cdot\,, Kv\rangle$ has a unique maximiser for each $v \in \overline{\mathscr{F}^w}$, [15, Theorem 5] implies that an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ Kv$ is equal to the unique maximiser. F may be alternately defined: for each $v \in \mathscr{F}^w$, $F(v) = \varphi_v \circ Kv$, where $\varphi_v : \mathbb{R} \to \mathbb{R}$ is an increasing function such that $\varphi_v \circ Kv \in \mathscr{F}$. By Lemma 3.7, F is a strongly continuous function.

By Theorem 2.50, the set of extreme points of $\overline{\mathscr{F}^w}$ is $\mathscr{F}$. Now, $\overline{\mathscr{F}^w}$ is bounded and convex, as Ω is a measure interval, by Theorem 2.60 (*vii*), so the set of extreme points is contained within the boundary. Hence, $\partial\overline{\mathscr{F}^w} \subseteq \mathscr{F}$. This inclusion only holds with equality when the base function f_0 is contant. It follows that the maximisers of the linear functionals $\langle\,\cdot\,, Kv\rangle$ lie in the set of rearrangements $\mathscr{F}$.

Let $(v_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $\overline{\mathscr{F}^w}$. The goal is to show that there is a convergent subsequence of $(F(v_n))_{n\in\mathbb{N}}$, since compactness and sequential compactness are equivalent in the metric space $L^p(\Omega)$, as shown in [62, Section 9.5 Theorem 16]. Theorem 2.60 (v) states that $\overline{\mathscr{F}^w}$ is weakly sequentially compact, so there exists a subsequence of $(v_n)_{n\in\mathbb{N}}$ that converges weakly to $v \in \overline{\mathscr{F}^w}$. For brevity, assume that $v_n \rightharpoonup v$. Define

$$\Psi : \overline{\mathscr{F}^w} \to \mathbb{R},\ \Psi(w) = \langle F(w), Kw\rangle\,.$$

It is the supremum of the linear function $\langle \cdot\,, Kw\rangle$ on the weakly compact and convex set $\overline{\mathscr{F}^w}$, so it easily follows

$$|\Psi(w) - \Psi(z)| \leq 2 \left\| f_0 \right\|_p \left\| Kw - Kz \right\|_q \text{ for all } w, z \in \overline{\mathscr{F}^w}.$$

The weak sequential compactness of Ψ follows from the compactness of K. Then $\lim_{n\to\infty} Kv_n = Kv$ strongly in $L^q(\Omega)$ and $\lim_{n\to\infty} \Psi(v_n) = \Psi(v)$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \langle F(v_n), Kv\rangle &= \lim_{n\to\infty} \left(\langle F(v_n), Kv_n\rangle + \langle F(v_n), Kv - Kv_n\rangle\right) \\
&= \lim_{n\to\infty} \left(\Psi(v_n) + \langle F(v_n), Kv - Kv_n\rangle\right) \\
&= \Psi(v) + 0 = \Psi(v).
\end{aligned}$$

Since $(F(v_n))_{n\in\mathbb{N}}$ is bounded, and it is deduced that $(F(v_n))_{n\in\mathbb{N}}$ is a maximising sequence for $\langle \cdot\ , Kv\rangle$. Therefore, [15, Theorem 2] proves $\lim_{n\to\infty} F(v_n) = F(v)$. This means that $F(\overline{\mathscr{F}^w})$ is sequentially compact, and so, a compact subset of $\mathscr{F}$. $\square$

The fixed points of this maximiser function are now of particular interest. Let $u \in \mathscr{F}$ satisfy $F(u) = u$. Then this fixed point u is characterised by the existence of an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $u = \varphi \circ Ku$. Setting $\psi = Ku$, it is obtained that $-\Delta\psi = u$, so

$$\left.\begin{aligned}
-\Delta\psi &= \varphi \circ \psi, \\
\psi &\in H_0^1(\Omega), \\
-\Delta\psi &\in \mathscr{F}
\end{aligned}\right\} .$$

The fixed points of F are precisely the solutions of the boundary value problem (5.1). The boundary value problem has been reduced to finding fixed points of a continuous function with compact image over a bounded and convex set.

Proposition 5.12. *Let $\Omega \subset \mathbb{R}^N$ be a non-empty open and bounded subset, and suppose $N \geq 2$, $N \in \mathbb{N}$ and $\frac{2N}{N+2} < p < \infty$. Let $f_0 > 0$ be a fixed function in $L^p(\Omega)$, and denote the set of rearrangements of f_0 on Ω by $\mathscr{F}$. Let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be the maximiser function given in Definition 5.10. Then F has a fixed point.*

Proof. $\overline{\mathscr{F}^w}$ is a non-empty, closed and convex subset of the normed vector space $L^p(\Omega)$, and $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ is a continuous function with compact image. By Theorem 4.12, F has a fixed point. $\square$

5.2 Nearest Point Maps and the Application of Degree Theory

It is not possible to directly apply Degree Theory to seek out fixed points of the maximiser function F, as the Leray-Schauder Degree involves considering open subsets of a Banach space. In this problem, the Banach space will be $L^p(\Omega)$. To overcome this hurdle, I use nearest point maps in uniformly convex spaces, as it is known that $L^p(\Omega)$ is a uniformly convex space.

Definition 5.13 (Uniformly Convex Space)**.** Let $(Y, \|\cdot\|)$ be a normed vector space. Y is a *uniformly convex space* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in Y$ with $\|x\| = \|y\| = 1$, it follows that: $\|x + y\| > 2 - \delta$ implies that $\|x - y\| < \varepsilon$.

J.A. Clarkson described the concept of a uniformly convex space in [18]. It should be noted that Clarkson's original formulation is the contrapositive of the above property.

Proposition 5.14. *A non-empty closed convex set C in a uniformly convex Banach space $(Y, \|\cdot\|)$ possesses a unique point closest to a given point. For every $y \in Y$, denote the closest point of C to y by $\mathscr{J} y$. Then $\mathscr{J} : Y \to C$ is a continuous operator.*

Proof. This is well-known, and may be found in [1, Problems 1.1.17(c) and (d)]. □

Theorem 5.15. *$L^p(\mu)$ is a uniformly convex Banach space for $1 < p < \infty$.*

Proof. This is the main result in [18]. □

Recall from Theorem 2.60 that $\overline{\mathscr{F}^w}$ is a non-empty closed convex set when the underlying measure space is a measure interval. Using nearest point maps, it is possible to construct a function from an open subset of $L^p(\Omega)$ into itself that shares fixed points with the maximiser function F.

Proposition 5.16. *Let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be defined as in Definition 5.10. Let*

$$\Gamma = \left\{ f \in L^p(\Omega) : \|f\|_p < 2 \|f_0\|_p \right\}.$$

Set

$$G : \Gamma \to \Gamma,\ G(v) = F(\mathscr{J} v),$$

where $\mathscr{J} : \Gamma \to \overline{\mathscr{F}^w}$ is the nearest point map. Then G is compact (that is, G is continuous with a compact image) and $Fix\ F = Fix\ G$.

Proof. The function G is continuous as the composition of two continuous functions. If $A \subseteq \Gamma$, then $\mathscr{J}(A) \subseteq \overline{\mathscr{F}^w}$, so that $G(A) = F(\mathscr{J}(A))$ is compact, by Proposition 5.11. It follows that G is compact.

If $F(v) = v \in \mathscr{F}$, then $G(v) = F(\mathscr{J} v) = F(v) = v$, as $\mathscr{J} v = v$. Conversely, if $G(v) = v$, then $v \in \mathscr{F}$, so that $F(v) = F(\mathscr{J} v) = G(v) = v$. Hence, $\mathrm{Fix}\, F = \mathrm{Fix}\, G$. $\square$

Since fixed points of G, and consequently of F, may be characterised as zeroes of the compact perturbation from the identity map $\mathrm{id} - G$, the Leray-Schauder degree may be applied to subsets $D \subset \Gamma$, and $d(\mathrm{id} - G, D, 0)$ is to be calculated for varied D. It is also necessary to consider how fixed points of the function F relate to one another.

Proposition 5.17. *Suppose (X, Σ, μ) is a finite measure space. Let $1 < p, q < \infty$ be conjugates, and let $S : L^p(X) \to L^q(X)$ be linear and continuous. Suppose $\mathscr{F}$ is the set of rearrangements of $f_0 : X \to \mathbb{R}$ such that the maximiser function*

$$F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}, \; F(v) = arg \max_{w \in \overline{\mathscr{F}^w}} \int_X wSv \; d\mu$$

is well-defined. Suppose $\sigma, \tau : X \to X$ are measure-preserving bijections such that $S(v \circ \sigma) = Sv \circ \tau$ for some $v \in \mathscr{F}$. Then $F(v \circ \sigma) = F(v) \circ \tau$.

Proof. I seek to show that $F(v \circ \sigma) \circ \tau^{-1}$ also maximises the integral $\int_X wKv \; \mathrm{d}\mu$ over $w \in \overline{\mathscr{F}^w}$. Consider the integral:

$$\begin{aligned}
\int_X F(v)Sv \; \mathrm{d}\mu &= \int_X (F(v) \circ \tau)(Sv \circ \tau) \; \mathrm{d}\mu && \text{as } \tau \text{ is measure-preserving} \\
&= \int_X (F(v) \circ \tau)S(v \circ \sigma) \; \mathrm{d}\mu && \text{by supposition} \\
&\leq \int_X F(v \circ \sigma)S(v \circ \sigma) \; \mathrm{d}\mu && \text{by definition of the maximiser function} \\
&= \int_X F(v \circ \sigma)(Sv \circ \tau) \; \mathrm{d}\mu && \text{by supposition} \\
&= \int_X (F(v \circ \sigma) \circ \tau^{-1})Sv \; \mathrm{d}\mu && \text{as } \tau^{-1} \text{ is measure-preserving} \\
&\leq \int_X F(v)Sv \; \mathrm{d}\mu && \text{by definition of the maximiser function.}
\end{aligned}$$

Hence, $F(v \circ \sigma) \circ \tau^{-1}$ maximises the integral, and since maximisers are unique, it must be that

$$F(v \circ \sigma) \circ \tau^{-1} = F(v),$$

that is,

$$F(v \circ \sigma) = F(v) \circ \tau,$$

as desired. $\square$

I introduce the concept of a *S-compatible measure-preserving bijection*, where S is a linear and continuous function between $L^p(X)$ and $L^q(X)$, where p, q are conjugate exponents, to codify the above result.

Definition 5.18 (S-compatible)**.** Let (X, Σ, μ) be a finite measure space, let $1 < p, q < \infty$ be conjugates and let $S : L^p(X) \to L^q(X)$ be a linear and continuous function. Let $\sigma : X \to X$ be a measure-preserving bijection. Then σ is *S-compatible* if

$$S(\zeta \circ \sigma) = S\zeta \circ \sigma \text{ for all } \zeta \in L^p(X).$$

It can be shown that the set of S-compatible measure-preserving bijections is a group under composition.

Lemma 5.19. *Let (X, Σ, μ) be a finite measure space, let $1 < p, q < \infty$ be conjugates and let $S : L^p(X) \to L^q(X)$ be a linear functional. The set of S-compatible measure-preserving bijections is a group under composition.*

Proof. I seek to show that the set of S-compatible measure-preserving bijections is a subgroup of the measure-preserving bijections between (X, Σ, μ) and itself. $\mathrm{id} : X \to X$ is certainly S-compatible. If $\sigma : X \to X$ is S-compatible, then

$$S(\zeta \circ \sigma) = S\zeta \circ \sigma \text{ for all } \zeta \in L^p(X)$$

may be rewritten as

$$S((\zeta \circ \sigma^{-1}) \circ \sigma) = S(\zeta \circ \sigma^{-1}) \circ \sigma \text{ for all } \zeta \in L^p(X).$$

It follows that

$$S(\zeta \circ \sigma^{-1}) = S(\zeta \circ \sigma^{-1}) \text{ for all } \zeta \in L^p(X),$$

so $\sigma^{-1} : X \to X$ is also S-compatible measure-preserving bijection. Finally, let $\sigma, \tau : X \to X$ be two S-compatible measure-preserving bijections, then for all $\zeta \in L^p(X)$:

$$S(\zeta \circ (\sigma \circ \tau)) = S((\zeta \circ \sigma) \circ \tau) = S(\zeta \circ \sigma) \circ \tau = S\zeta \circ (\sigma \circ \tau).$$

In conclusion, $\sigma \circ \tau : X \to X$ is a S-compatible measure-preserving bijection, and the set of S-compatible measure-preserving bijections is a subgroup of the measure-preserving bijections, and therefore, a group in its own right. $\square$

Now, I turn back to the linear inverse to the weak negative Laplacian operator. A K-compatible measure-preserving bijection, for the linear functional under consideration, is the rotation around a point, where the domain is rotationally symmetric.

Lemma 5.20. *Suppose $\Omega \subset \mathbb{R}^N$ is a non-empty open and bounded domain which is rotationally symmetric, for some $N \in \mathbb{N}$. Let $R : \Omega \to \Omega$ be a rotation such that $R(\Omega) = \Omega$. Suppose $K : L^p(\Omega) \to H_0^1(\Omega)$ is the inverse of the weak negative Laplacian with zero Dirichlet boundary conditions, where $1 < p < \infty$. Then*

$$K(\zeta \circ R) = K\zeta \circ R \text{ for all } \zeta \in L^p(\Omega),$$

that is, $R : \Omega \to \Omega$ is a K-compatible measure-preserving bijection.

Proof. For brevity, I will use R to also denote the rotation matrix. Note that R is both constant and orthogonal.
Let $\zeta \in L^p(\Omega)$. It holds that

$$-\Delta[K(\zeta \circ R)] = \zeta \circ R,$$

but also by Faà di Bruno's formula, for $x \in \Omega$:

$$\begin{aligned}
-\Delta[K\zeta \circ R](x) &= -\sum_{i=1}^{N} D_i D_i [K\zeta \circ R](x) \\
&= -\sum_{i=1}^{N} \left[\sum_{j=1} D_j K\zeta(Rx) \frac{\partial^2 R}{\partial x_i \partial x_i} + \sum_{j,l=1}^{N} D_j D_l K\zeta(Rx) R_{ji} R_{li} \right] \\
&= -\sum_{i,j,l=1}^{N} D_j D_l K\zeta(Rx) R_{ji} R_{li} \\
&= -\sum_{j,l=1}^{N} D_{jl} K\zeta(Rx) \left(\sum_{i=1}^{N} R_{ji} R_{li} \right) \\
&= -\sum_{j,l=1}^{N} D_{jl} K\zeta(Rx) \delta_{jl} \\
&= -\sum_{j=1}^{N} D_j D_j K\zeta(Rx) \\
&= \zeta(Rx) = (\zeta \circ R)(x).
\end{aligned}$$

Since $(K\zeta \circ R)|_{\partial\Omega} = K(\zeta \circ R)|_{\partial\Omega} = 0$, and solutions to the Dirichlet problem has unique solutions, so

$$K\zeta \circ R = K(\zeta \circ R).$$

By Definition 5.18, $R : \Omega \to \Omega$ is a K-compatible measure-preserving bijection. $\square$

The above lemma is used to demonstrate that rotations preserve both fixed points of F and the value of the Leray-Schauder degree.

Theorem 5.21. *Let $\Omega \subset \mathbb{R}^N$, where $N \in \mathbb{N}$, be a domain which is rotationally symmetric, with respect to a rotation $R : \Omega \to \Omega$. Let $1 < p, q < \infty$ be conjugates, and suppose $\mathscr{F}$ denote the set of rearrangements of $f_0 \in L^p(\Omega)$. Suppose $K : L^p(\Omega) \to L^q(\Omega)$ is a linear functional, and that the maximiser of the integral $\int_\Omega wKv \; d\mu$ over $w \in \overline{\mathscr{F}^w}$ for each fixed $v \in \overline{\mathscr{F}^w}$ is unique. Let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be the corresponding maximiser function, and set $\Gamma = \left\{\zeta \in L^p(\Omega) : \|\zeta\|_p < 2\|f_0\|_p\right\}$. Let $\mathscr{J} : L^p(\Omega) \to \overline{\mathscr{F}^w}$ be the nearest point map, and*

$$G : \Gamma \to \Gamma, G(v) = F(\mathscr{J} v).$$

Let $D \subseteq \Gamma$, and denote $D \circ R = \{v \circ R : v \in D\}$. Then

(i) $v \in \mathscr{F} \Longleftrightarrow v \circ R \in \mathscr{F}$,

(ii) $v \in \overline{\mathscr{F}^w} \Longleftrightarrow v \circ R \in \overline{\mathscr{F}^w}$,

(iii) $F(v) = v \Longleftrightarrow F(v \circ R) = v \circ R$,

(iv) $d(id - G, D, 0) = d(id - G, D \circ R, 0)$,

(v) *Let $z \in \Gamma$, then* $d(id - G, D, z) = d(id - G, D, z \circ R)$.

Proof. Both (i) and (ii) should be clear from Definition 1.1.
(iii) By Lemma 5.20, R is a K-compatible measure-preserving bijection. By Lemma 5.19, R^{-1} is also a K-compatible measue-preserving bijection. This can be elaborated as

$$K(\zeta \circ R) = K\zeta \circ R \text{ and } K(\zeta \circ R^{-1}) = K\zeta \circ R^{-1} \text{ for all } \zeta \in L^p(\Omega).$$

Let $v \in \overline{\mathscr{F}^w}$ and $F(v) = v$. Invoking Proposition 5.17, setting $\sigma = \tau = R$, it follows that

$$F(v \circ R) = F(v) \circ R = v \circ R.$$

Conversely, suppose $F(v \circ R) = v \circ R$, then by Proposition 5.17 again:

$$F(v) = F(v \circ (R \circ R^{-1})) = F(v \circ R) \circ R^{-1} = (v \circ R) \circ R^{-1} = v.$$

(iv) Note that R can be represented by the matrix (R_{ij}). Let $\psi \in C^1(\Gamma) \cap C(\overline{\Gamma})$ be sufficiently close to $\mathrm{id} - G$ to ensure that they have the same degree over D and $D \circ R$ with respect to 0, and have finite dimensional image, as in Definition 4.6. Denote $y = Rx$. Then

$$\begin{aligned}\frac{\partial(\phi \circ R)_i(x)}{\partial x_j} &= \sum_{l=1}^{N} \frac{\partial \phi_i(y)}{\partial y_l} \frac{\partial y_l}{\partial x_j} \\ &= \sum_{l=1}^{N} \frac{\partial \phi_i(x)}{\partial x_l} R_{lj}.\end{aligned}$$

This means that

$$\left(\frac{\partial(\phi \circ R)_i}{\partial x_j}\right) = \left(\frac{\partial \phi_i}{\partial x_j}\right) R.$$

Hence:

$$J_{\phi \circ R}(x) = J_\phi(x) \det(R) = J_\phi(x).$$

Now,

$$\begin{aligned} d(\mathrm{id} - G, D, 0) &= d(\phi, D, 0) && \text{by construction} \\ &= \sum_{\zeta \in \phi^{-1}(0)} \operatorname{sgn}\, J_\phi(\zeta) && \text{by Brouwer degree definition} \\ &= \sum_{\zeta \in (\phi \circ R)^{-1}(0)} \operatorname{sgn} J_{\phi \circ R}(\zeta) && \text{by prior calculation} \\ &= d(\phi, D \circ R, 0) && \text{by Brouwer degree definition} \\ &= d(\mathrm{id} - G, D \circ R, 0) && \text{by construction.} \end{aligned}$$

Note, (v) is a simple generalisation of (iv). This follows from the consideration that $\zeta \circ \phi^{-1}(z)$ if, and only if, $\zeta \circ (\phi \circ R)^{-1}(z \circ R)$. $\square$

There are other scenarios where the local degree of the maximiser function can be quickly calculated.

Proposition 5.22. *Suppose $\Omega \subset \mathbb{R}^N$ is non-empty and bounded, for some $N \in \mathbb{N}$, where $\partial\Omega$ is a $C^{1,1}$ boundary. Suppose $\frac{N}{2} < p < \infty$ and let q be the conjugate exponent of p, $\mathscr{F}$ is the set of rearrangements of a measurable function $f_0 \in L^p(\Omega)$, $\Gamma = \{\zeta \in L^p(\Omega) : \|\zeta\| \leq 2\|f_0\|\}$, and $G : \Gamma \to \Gamma$ is the maximiser function, which is assumed to be well-defined and is defined in Proposition 5.16. Suppose $K : L^p(\Omega) \to L^q(\Omega)$ is the linear inverse of the weak negative Laplacian operator, as defined in Definition 5.2. Denote by Ψ the energy function over $\overline{\mathscr{F}^w}$, and u is an isolated fixed point of G.*

Suppose there exists a $\delta > 0$ such that

$$\int_\Omega G\zeta K u \ d\mu \geq \int_\Omega \zeta K \zeta \ d\mu \textit{ for all } \zeta \in \mathbb{B}(u, \delta) \cap \overline{\mathscr{F}^w}.$$

Then u is a fixed point of index 1 for G.

Proof. I have assumed that u is an isolated fixed point of G. Applying $\zeta = u$ in the above inequality shows that u is a local maximiser of the energy functional Ψ.
Since G is continuous, and there is a local ball around u, say B', where u is the local maximiser of Ψ, there exists a $0 < \eta < \delta$ that is sufficiently small to imply

$$f \in \overline{\mathbb{B}(u, \eta)} \cap \overline{\mathscr{F}^w} \text{ implies } \Psi(Gf) \leq \Psi(u).$$

It holds that, as in [28], for $f \in L^p(\Omega)$:

$$\begin{aligned}
\Psi(G\zeta) - \Psi(\zeta) &= \frac{1}{2} \int_\Omega (GfK(Gf) - fKf) \ \mathrm{d}\mu \\
&= \frac{1}{2} \int_\Omega ((Gf - f)K(Gf - f) + 2(Gf - f)Kf) \ \mathrm{d}\mu \\
&= \frac{1}{2} \int_\Omega (Gf - f)K(Gf - f) \ \mathrm{d}\mu + \int_X (Gf - f)Kf \ \mathrm{d}\mu \\
&\geq 0 \qquad\qquad \text{by definition of } G.
\end{aligned}$$

Hence, for $f \in \overline{\mathbb{B}(u, \eta)} \cap \overline{\mathscr{F}^w} \setminus \{u\}$, it holds that $\Psi(f) \leq \Psi(Gf) < \Psi(u)$. Set $B = \mathbb{B}(u, \eta)$, then $d(\mathrm{id} - G, B, 0)$ can be calculated.

Assume, seeking contradiction, that there exists $f \in \partial B$ and $t \in [0, 1]$ such that

$$f = tG(f) + (1 - t)u,$$

so $f \in \overline{\mathscr{F}^w}$. f is not a fixed point of G or equal to u, so $t \in (0, 1)$. It follows

$$\begin{aligned}
\Psi(f) &= t^2 \Psi(Gf) + 2t(1 - t) \left(\frac{1}{2} \int_\Omega G(f)Kf \ \mathrm{d}\mu \right) + (1 - t)^2 \Psi(u) \\
&> t^2 \Psi(f) + 2t(1 - t)\Psi(f) + (1 - t)^2 \Psi(f) \qquad\qquad \text{by supposition} \\
&= \Psi(f).
\end{aligned}$$

This is a contradiction. The homotopy between $\mathrm{id} - u$ and $\mathrm{id} - G$ is valid over $\overline{B}$, and by homotopy invariance of the Leray-Schauder degree:

$$d(\mathrm{id} - G, B, 0) = d(\mathrm{id} - u, B, 0) = d(\mathrm{id}, B, u) = 1.$$

The result is complete. $\square$

Finally, I show that properties of the weak closure of the set of rearrangements are inherited by subsets, which will be used in the main proof.

Lemma 5.23. *Suppose (X, Σ, μ) is a measure interval, $1 \leq p < \infty$, $X_0 \subset X$ is a measurable subset, $\mathscr{F}$ is a set of rearrangements on X and $g_0 : X_0 \to \mathbb{R}$ is a measurable function with $g_0 \in L^p(X)$. Then the set*

$$\mathscr{R} = \overline{\{g \in \mathscr{F} : g|_{X_0} = g_0\}}^w \subset L^p(X)$$

is convex, where $\mathscr{R}$ inherits the topology from $L^p(X)$.

Proof. If there is no such $g \in \mathscr{F}$ that satisfies $g|_{X_0} = g_0$, then $\mathscr{R}$ is empty, and the result follows.

Suppose $\mathscr{R}$ is non-empty, and let $f, g \in \mathscr{F}$ satisfy $f|_{X_0} = g|_{X_0} = g_0$. By Theorem 1.20, it follows that $f|_{X \setminus X_0}$ and $g|_{X \setminus X_0}$ are rearrangements. By Theorem 2.60 (vii), the weak closure of $\{g \in \mathscr{F} : g|_{X_0} = g_0\}$ is convex, and so $\mathscr{R}$ is convex.

Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\mathscr{F}$ where $f_n|_{X_0} = g_0$ μ-a.e. and $f_n \rightharpoonup f$. Assume, seeking contradiction, that there exists a measurable subset of X_0 with positive measure such that $f \neq g_0$ on this set. Without loss of generality, let

$$E = \{x \in X_0 : f(x) > g_0(x)\}.$$

Consequently, set the linear functional

$$\varphi : L^p(X) \to \mathbb{R}, \ \varphi(g) = \int_E g \ \mathrm{d}\mu.$$

Then

$$\begin{aligned}
\int_E g \ \mathrm{d}\mu &= \varphi(g) \\
&= \varphi\left(\lim_{n \to \infty} f_n\right) \\
&= \lim_{n \to \infty} \varphi(f_n) && \text{by linearity} \\
&= \varphi(f) && \text{by weak convergence} \\
&= \int_E f \ \mathrm{d}\mu.
\end{aligned}$$

It is a contradiction for these integrals to be equal, since E is precisely the subset of X_0 where the value of f is strictly greater than the value of g_0. Therefore, no such measurable set exists. Thus, $f|_{X_0} = g_0$ μ-a.e, and so $f \in \overline{\mathscr{F}^w} \cap \{g \in L^p(X) : g|_{X_0} = g_0\}$. Hence,

$$\mathscr{R} \subseteq \overline{\mathscr{F}^w} \cap \{g \in L^p(X) : g|_{X_0} = g_0\}.$$

Since the other inclusion is clear, then $\mathscr{R} = \overline{\mathscr{F}^w} \cap \{g \in L^p(X) : g|_{X_0} = g_0\}$. The two sets, $\overline{\mathscr{F}^w}$ and $\{g \in L^p(X) : g|_{X_0} = g_0\}$, are convex, so their intersection is convex. $\square$

5.3 The Main Results

For convenience, I repeat the rules governing the Leray-Schauder degree, which will be used repeatedly throughout this section.

Theorem 5.24 (Rules governing the Leray-Schauder Degree)**.** *Suppose $(X, \|\cdot\|)$ is a normed vector space, with an open, bounded subset $D \subset X$. Let $\phi, \psi, id \in \mathscr{K}_1(\overline{D}) = \{id - \eta : \eta \text{ is compact}\}$ and $z \in X$, where id is the identity map. Let $H(t)$ be a homotopy of compact transformations on $\overline{D}$. Let $\phi_t = id - H(t)$ and suppose $d(\phi, D, z)$ is the Leray-Schauder degree. Then*

(d1) if $z \in D$, then $d(id, D, z) = 1$; if $z \notin \overline{D}$, then $d(id, D, z) = 0$;

(d2) if $d(\phi, D, z) \neq 0$, then there is $x \in D$ such that $\phi(x) = z$;

(d3) if $z \notin \phi_t(\partial D)$ for all $t \in [0, 1]$, then $d(\phi_t, D, z)$ is independent of $t \in [0, 1]$;

(d4) if $z \notin \phi(\partial D)$ and $q \in X$, and $\psi(x) = \psi(x) - q$ for all $x \in \overline{D}$, then $d(\phi, D, z) = d(\psi, D, z - q)$;

(d5) if D is the disjoint union of open sets $D_1, \ldots, D_n$ for some $n \in \mathbb{N}$, then

$$d(\phi, D, z) = \sum_{i=1}^{n} d(\phi, D_i, z).$$

The first stage of the main result is to calculate the number of fixed points of the maximiser function for the simplest problem: the domain is a single ball. This requires a result by Fraenkel.

Theorem 5.25. *Let $\Omega \subset \mathbb{R}^N$ be non-empty, open and bounded for some $N \in \mathbb{N}$. Suppose Ω is connected and Steiner symmetric relative to the hyperplane*

$$T_0 = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^N : x_1 = 0\right\}.$$

Also, define $M := \sup\{x_1 : x \in \Omega\}$ and $Z(\mu) = \{x \in \Omega : x_1 > \mu\}$. Assume that $v : \overline{\Omega} \to \mathbb{R}$ has the following properties:

(i) $v \in C(\overline{\Omega}) \cap C^1(\Omega),\ v > 0$ *in* Ω, $v|_{\partial\Omega} = 0$;

(ii) for all $\xi \in C_c^\infty(\Omega),\ \int_\Omega (-\Delta\xi v + \xi\varphi(v))\ d\mu = 0$,
where $\varphi : [0,\infty) \to \mathbb{R}$ *has a decomposition* $\varphi = \varphi_1 + \varphi_2$ *such that* $\varphi_1 : [0,\infty) \to \mathbb{R}$ *is locally Lipschitz continuous,* $\varphi_2 : [0,\infty) \to \mathbb{R}$ *is non-decreasing and* φ_2 *is identically* 0 *on the compact interval* $[0,\kappa]$ *for some* $\kappa > 0$.

Then the following conclusions hold:

$$v(z) < v(z^\beta) \text{ if } \beta \in (0, M) \text{ and } z \in Z(\beta),$$
$$\partial_1 v(x) < 0 \text{ if } x_1 > 0 \text{ and } x \in \Omega.$$

Proof. This result is proved in Fraenkel's book [34, p. 88]. □

Corollary 5.26. *If in the above theorem, the set* Ω *is a ball, say*

$$\Omega = \mathbb{B}(0,\alpha) = \{y \in \mathbb{R}^N : |y| < \alpha\} \subset \mathbb{R}^N,$$

then v *is spherically symmetric, that is,* v *depends only on* $r = |x|$, *and* $\frac{dv}{dr} < 0$ *for* $0 < r < \rho$.

Proof. This result is [34, p. 92]. □

Now, the number of fixed points for the simplest problem can be calculated.

Lemma 5.27 (Domain is an Open Ball)**.** *Let* $\Omega = \mathbb{B}(0,\rho) \subset \mathbb{R}^N$, *where* $\rho > 0$ *and* $N \geq 2,\ N \in \mathbb{N}$. *Consider the maximiser function* $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$, *where* $\mathscr{F}$ *is the set of all rearrangements of* $f_0 > 0$, *where* $f_0 \in L^p(\Omega)$. *Suppose further that* $\frac{2N}{N+2} < p < \infty$ *and* $f_0 \geq k > 0$ *almost everywhere, where* k *is the essential infimum of* f_0, *and is attained on a set of nonzero Lebesgue measure. Then in this case,* F *has only one fixed point, namely the symmetric-decreasing rearrangement of* f_0 *on* Ω.

Proof. By Proposition 5.12, it is already known that one fixed point exists. Uniqueness of this fixed point shall be demonstrated. Consider a fixed point of F, say u, and write $\psi = Ku$. By supposition, $\varphi \circ \psi \geq k$ almost everywhere, and $\varphi \circ \psi = k$ on a set S of positive measure. Note that level sets of ψ have zero measure, the essential range of ψ on S must have more than one point. Since φ is non-decreasing, ψ is positive, and k is a value of $\varphi \circ \psi$, there must exists a number $\kappa > 0$ such that $\varphi(s) = k$ for $0 < s < \kappa$ and $\varphi(s) > k$ for $s \geq k$.

It can be seen that φ may be decomposed in the following manner: $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \equiv k$, which is constant and hence Lipschitz continuous,and $\varphi_2 = \varphi - k$, which is non-decreasing and vanishes on the interval $[0, \kappa]$. The non-decreasing function φ now satisfies the conditions given in Theorem 5.25. It follows that ψ is spherically symmetric and radially decreasing. It also holds that $u = -\Delta\psi = \varphi \circ \psi$ is spherically symmetric and radially decreasing, and u is a rearrangement of f_0. Only one such function exists, so the fixed point must be unique. $\square$

Even for this simple problem, it is worth calculating what the local degree of the unique fixed point discovered by Lemma 5.27 is.

Proposition 5.28. *Let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be defined as in Definition 5.10. Let $1 < p < \infty$ and*

$$\Gamma = \left\{ f \in L^p(\Omega) : \|f\|_p < 2\|f_0\|_p \right\}.$$

Set

$$G : \Gamma \to \Gamma,\ G(v) = F(\mathscr{J} v),$$

where $\mathscr{J} : \Gamma \to \overline{\mathscr{F}^w}$ is the nearest point map. Then $d(id - G, \Gamma, 0) = 1$.

Proof. Note that for all $f \in L^p(\Omega)$, it holds that $G(f) = F(\mathscr{J} f) \in \mathscr{F}$, which implies that $\|G(f)\|_p = \|f_0\|_p$. Consider the homotopy between id and id $- G$, where

$$H : \overline{\Gamma} \times [0,1] \to L^p(\Omega),\ H(t, f) = f - tG(f).$$

This is a homotopy of compact perturbations from the identity function.

Assume, seeking contradiction, that there exists a $t \in [0,1]$ such that $0 \in [\mathrm{id} - tG](\partial\Gamma)$. Hence, there exists $f \in \partial\Gamma = \left\{ g \in L^p(\Omega) : \|g\|_p = 2\|f_0\|_p \right\}$ such that $f - tG(f) = 0$. Taking norms yields

$$tG(f) = f \text{ which implies } t\|f_0\|_p = 2\|f_0\|_p.$$

Clearly, $\|f_0\|_p > 0$, so $t = 2$. Since $t \in [0,1]$, this is absurd.
This means that the homotopy H is valid. Then

$$\begin{aligned} d(\mathrm{id} - G, \Gamma, 0) &= d(\mathrm{id}, \Gamma, 0) && \text{by } (d3) \\ &= 1 && \text{by } (d1) \end{aligned}$$

$\square$

Remark 5.29. In the case of Lemma 5.27, the local degree of the unique fixed point u is equal to the degree of $d(\mathrm{id} - G, U, 0)$ for every open subset $U \subseteq \Gamma$ such that $u \in U$ and U contains no other fixed points. Since the fixed point u is unique, U may be set to Γ, so the local degree of u is 1. This is true for all compact maps that have only one fixed point.

Prior to the main result, which will involve the consideration of a domain approximately equal to the union of n balls in $\mathbb{R}^N$, I require a selection of preparatory lemmata.

Lemma 5.30. *Let $(Y, +, \cdot, \|\cdot\|)$ be a normed vector space, let C be a closed convex subset of Y, and $\beta > 0$. If $S \subset C$ is nonempty, then the boundary relative to C of the set $\{y \in C : \operatorname{dist}(y, S) < \beta\}$ is equal to $\{y \in C : \operatorname{dist}(y, S) = \beta\}$.*

Proof. Denote the two sets

$$U = \{y \in C : \operatorname{dist}(y, S) < \beta\} \text{ and } V = \{y \in C : \operatorname{dist}(y, S) = \beta\}.$$

Define the function

$$\psi : C \to \mathbb{R},\ \psi(y) = \operatorname{dist}(y, S).$$

It is well-known that this function ψ is Lipschitz continuous with constant 1, as $S \neq \emptyset$, so U is an open set in C.

Let $x \in V$ be an arbitrary point, then it is sought that $x \in V$ is the limit of a sequence in $U \cap C$. By definition, there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in S such that $\lim_{n\to\infty} \|x - s_n\| = \beta$. Now, set

$$t_n = \frac{n\beta}{(n+1)\|x - s_n\|} \in (0, 1) \text{ for all } n \in \mathbb{N}.$$

Since $\|x - s_n\| \geq \beta > 0$, this sequence is well-defined. By the algebra of limits, $\lim_{n\to\infty} t_n = 1$. Let

$$x_n = t_n x + (1 - t_n)s_n \text{ for all } n \in \mathbb{N}.$$

By the convexity of C, $x_n \in C$ for all $n \in \mathbb{N}$. Then, it follows

$$\|x_n - s_n\| = t_n \|x - s_n\| = \frac{n\beta}{n+1} < \beta \text{ for all } n \in \mathbb{N},$$

so $x_n \in U$ and $\lim_{n\to\infty} x_n = x$ in the norm of Y. This shows $V \subseteq \overline{U}$,and since $V \subseteq C \backslash U$, it is deduced that $V \subseteq \partial U$.

Conversely, it is noted that

$$\partial U \cap C \subseteq \{y \in C : \operatorname{dist}(y, S) \leq \beta\}$$

since the latter set is closed and contains U. Since U is open, it is clear that $\partial U \cap U = \emptyset$. Hence, it must be that $\partial U = V$. $\square$

Next, the consistency of the numerous constraints on the domain and function must be proven, and this is encapsulated in the following three results.

Denote, for an open and bounded subset $D \subset \mathbb{R}^N$, where $N \in \mathbb{N}$ and $p^{-1} + q^{-1} = 1$, the linear inverse of the negative Laplacian with zero Dirichlet conditions on the set D:

$$K_D : L^p(D) \to L^q(D), \; -\Delta K_D(\zeta) = \zeta \text{ and } K_D(\zeta)|_{\partial D} = 0 \text{ for all } \zeta \in L^p(D).$$

Lemma 5.31. *Suppose there are real numbers $0 < r < r_1 \leq \ldots \leq r_n$ for some $n \in \mathbb{N}$; suppose $0 < \lambda < \mu(\mathbb{B}(0,r))^{\frac{1}{q}}$, where $1 < p, q < \infty$ are conjugates, that is, $p^{-1} + q^{-1} = 1$; set $D_i = \mathbb{B}(0, r_i) \subset \mathbb{R}^N$ for all $i \in \{1, \ldots, n\}$ and some $N \in \mathbb{N}$. Let*

$$\alpha_i = \inf \left\{ K_{\mathbb{B}(0,r_i)} f(x) : x \in \mathbb{B}(0,r), \; f \in L^p(D_i), \; f \geq 0 \text{ a.e.}, \; \|f\|_p \leq 1, \; \int_{D_i} f \; d\mu \geq \lambda \right\}.$$

Then the infimum in the formula for each α_i is attained, so $\alpha := \min \{\alpha_1, \ldots, \alpha_n\} > 0$.

Proof. Each set of functions over which the infima are taken are closed, bounded, convex and non-empty, as each set contains a constant function. By weak compactness, each infimum must be attained by a non-trivial function, and is therefore positive. $\square$

Lemma 5.32. *Let $0 < r < r_1 \leq \ldots \leq r_n$ be given for some $n \in \mathbb{N}$, and let $D_1, \ldots, D_n \subset \mathbb{R}^N$ be balls with disjoint closures having respective radii $r_1, \ldots, r_n$, with some $N \in \mathbb{N}$. Suppose $1 < p, q < \infty$ satisfy $p^{-1} + q^{-1} = 1$, $\frac{N}{2} < p$ and let $0 < \lambda < \mu(D_1)^{\frac{1}{q}}$, and let $\alpha > 0$ be the positive number provided by Lemma 5.31. Let V_0 be a right circular cone with vertex 0. Then there exists a positive number $\varepsilon > 0$ such that if $\Omega \subset \mathbb{R}^N$ is an open set satisfying*

(B1) $\overline{D_i} \subseteq \Omega$ *for* $i \in \{1, \ldots, n\}$,

(B2) $\operatorname{diam} \Omega \leq \operatorname{diam} \left(\bigcup_{i=1}^n D_i\right) + 1$,

(B3) $\partial\Omega$ *is of class* C^2 *and has the uniform exterior cone property with respect to* V_0, *that is,*

$$\text{for all } x \in \partial\Omega \text{ there exists a rotation } \theta \text{ such that } (x + \theta V_0) \cap \overline{\Omega} = \{0\},$$

and

(B4) for all $x \in \Omega \setminus \bigcup_{i=1}^n D_i$, *we have that* $\operatorname{dist}(x, \partial\Omega) < \varepsilon$,

then it holds

$$K_\Omega f(x) \leq \frac{\alpha}{4} \|f\|_p \text{ for all } x \in \Omega \text{ such that } \operatorname{dist}(x, \partial\Omega) < \varepsilon \text{ and for all } f \in L^p(\Omega).$$

Proof. Consider an open set Ω satisfying the conditions $(B1)-(B3)$. Then Ω lies inside a ball $D = \mathbb{B}(0, R_1)$ of radius $R_1 = \operatorname{dist}(\bigcup_{i=1}^{n} D_i) + 1$. From a consideration of elliptic regularity theory, it follows for $f \in L^p(D)$:

$$\|K_D f\|_{\sup} \leq c_0 \|f_0\|_p$$

where the positive constant $c_0 > 0$ depends only on R_1 and p. A consequence of the Maximum Principle is that

$$\|K_\Omega f\|_{\sup} \leq c_0 \|f\|_p .$$

For $x_0 \in \partial\Omega$ and $\varepsilon > 0$, it is immediate from regularity theory [36, Theorems 8.27 and 8.22] that:

$$\begin{aligned} \sup_{x \in \Omega \cap \mathbb{B}(x_0, \varepsilon)} |K_\Omega f(x)| &\leq c_1 \varepsilon^\gamma \left(\|K_\Omega f\|_{\sup} + \|f\|_p \right) \\ &\leq c_1 \varepsilon^\gamma (c_0 + 1) \|f\|_p \end{aligned}$$

where the two positive constants $c_1, \gamma > 0$ depend only on R_1, p and V_0. Now, set

$$\varepsilon = \left(\frac{\alpha}{4c_1(c_0+1)} \right)^{\frac{1}{q}} > 0.$$

If Ω also satisfies $(B4)$, then for all $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < \varepsilon$:

$$\begin{aligned} |K_\Omega f(x)| &\leq c_1 \varepsilon^\gamma (c_0 + 1) \|f\|_p \\ &= c_1(c_0+1) \frac{\alpha}{4c_1(c_0+1)} \|f\|_p \\ &= \frac{\alpha}{4} \|f\|_p . \end{aligned}$$

This completes the result. $\square$

Lemma 5.33. *Let $R_1 > 0$ and $n \in \mathbb{N}$. Let $\Omega \subset \mathbb{R}^N$ be a C^2 domain contained in a ball of radius R_1 and let $\alpha_0 > 0$. Then there is a constant $k_0 > 0$, chosen to be independent of Ω, such that*

$$0 \leq K_\Omega(k_0 \mathbb{1}_\Omega) < \frac{\alpha_0}{4}$$

throughout Ω.

Proof. Let D be an open ball of radius R_1, and let the constant k_0 satisfy

$$0 < k_0 < \frac{\alpha_0}{4 \|K_D \mathbb{1}_D\|_{\sup}},$$

that is,

$$k_0 \left\| K_D \mathbb{1}_D \right\|_{\sup} < \frac{\alpha_0}{4}.$$

If Ω is an open subset of D, then by the Maximum Principle, it is obtained that

$$K_\Omega \mathbb{1}_\Omega \leq K_D \mathbb{1}_\Omega \leq K_D \mathbb{1}_D$$

throughout Ω. Thus, $k_0 > 0$ has the desired properties. $\square$

Finally, the main result supposes that there is a domain that satisfies $(B1) - (B4)$ of Lemma 5.31 and has a positive infimum less than k_0 as defined in Lemma 5.33.

Theorem 5.34. *Let $0 < r < r_1 \leq \ldots \leq r_n$ and $0 < \lambda < \frac{1}{n}\mu(\mathbb{B}(0, r_1))^{\frac{1}{q}}$ be given, where $n \in \mathbb{N}$, $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, $p > \frac{N}{2}$ and $N \geq 2,\ N \in \mathbb{N}$. Let $\alpha = \alpha_0$, which is provided by Lemma 5.31. Let $B_1, \ldots, B_n \subset \mathbb{R}^N$ be balls with disjoint closures having respective radii $r_1, \ldots, r_n$, let V_0 be a right circular cone with vertex 0, let $\varepsilon > 0$ be the number provided by Lemma 5.32, and let $k_0 > 0$ be the positive number provided by Lemma 5.33.*

Suppose Ω is a domain satisfying the conditions $(B1) - (B4)$ in Lemma 5.32, let $0 \leq f_{00} \in L^p(\Omega)$ be a function vanishing outside of a set of measure $\mu(\mathbb{B}(0, r))$ and satisfying the integral condition

$$0 < \lambda n \left\| f_{00} \right\|_p < \int_\Omega f_{00}\ d\mu.$$

Let $0 < k \leq k_0$ and set

$$f_0 : \Omega \to \mathbb{R},\ \ f_0(x) = k + f_{00}(x).$$

Let $\mathscr{F} \subset L^p(\Omega)$ be the set of rearrangements of f_0 on Ω, and let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be the maximiser function defined in Definition 5.10. Then F has at least $2^n - 1$ fixed points and they all belong to $\mathscr{F}$.

Proof. Assume, without loss of generality, that $\left\| f_{00} \right\|_p = 1$ and that F has a finite number of fixed points. If this second assumption was contravened, the result would be achieved.

I shall begin by setting notation. Let $\mathscr{P}$ denote the set of non-empty subsets of $\{1, \ldots, n\}$, $Z \in \mathscr{P}$, let $Z \in \mathscr{P}$ and set $B_Z = \bigcup_{i \in Z} B_i$. Then consider the weak closure of a subset of rearrangements:

$$\mathscr{R}_Z := \overline{\{g \in \mathscr{F} : g^{-1}(k, \infty) \subseteq B_Z\}^w} \text{ for } Z \in \mathscr{P},$$

where the notation means $g^{-1}(k,\infty)$ is a subset of B_Z up to sets of zero Lebesgue measure. Note that $g^{-1}(k,\infty) \subseteq B_Z$ up to sets of Lebesgue measure zero if, and only if, $g(x) = k$ for almost all $x \in \Omega \setminus B_Z$. Since the inequality $f_0 > k$ holds for a measurable set with measure less than $\mu(\mathbb{B}(0,r))$, then $\mathscr{R}_Z$ is non-empty for each $Z \in \mathscr{P}$. By Lemma 5.23, for every $Z \in \mathscr{P}$,

$$\mathscr{R}_Z = \overline{\left\{g \in \mathscr{F} : g|_{\Omega \setminus B_Z} = k\right\}}^w$$

and each $\mathscr{R}_Z$ is convex.

It must be demonstrated that for $W, Z \in \mathscr{P}$ with $W \cap Z = \emptyset$, it holds that $\mathscr{R}_W \cap \mathscr{R}_Z = \emptyset$. Assume, seeking contradiction, that there exists a function lying in the intersection $f \in \mathscr{R}_W \cap \mathscr{R}_Z$. Thus, there exists sequences $(f_m)_{m\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$ such that

$$f_m^{-1}((k,\infty)) \subset B_W \text{ and } g_m^{-1}((k,\infty)) \subset B_Z \text{ for all } m \in \mathbb{N}$$

and $f_m \rightharpoonup f$ and $g_m \rightharpoonup f$. Define

$$\Phi_W : L^p(\Omega) \to \mathbb{R},\ \Psi_W(\zeta) = \int_{B_W} \zeta \ \mathrm{d}\mu.$$

Then, for each $m \in \mathbb{N}$,

$$\begin{aligned}\Psi_W(f_m) &= \int_{B_W} f_m \ \mathrm{d}\mu = \int_\Omega f \ \mathrm{d}\mu - k \sum_{i \notin W} \mu(B_i),\\ \Psi_W(g_m) &= \int_{B_W} g_m \ \mathrm{d}\mu = k \sum_{i \in W} \mu(B_i).\end{aligned}$$

Both of the sequences $(\Phi_W(f_m))_{m\in\mathbb{N}}$ and $(\Phi_W(g_m))_{m\in\mathbb{N}}$ are constant sequences, of distinct values, since $\mu(\{x \in \Omega : f_0 > k\}) > 0$. Since $f_m \rightharpoonup f$ and $g_m \rightharpoonup f$, this is a clear contradiction. Hence, for $W, Z \in \mathscr{P}$, $W \cap Z = \emptyset$ implies that $\mathscr{R}_W \cap \mathscr{R}_Z = \emptyset$.

Since $p > \frac{N}{2}$ and the domain Ω satisfies an exterior cone codition at each point of $\partial\Omega$, [36, Corollary 9.18] means that $Kv \in W^{2,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ for each $v \in L^p(\Omega)$. Also, set

$$\Gamma = \left\{f \in L^p(\Omega) : \|f\|_p < 2\|f_0\|_p\right\}.$$

Recall that the function $G : \Gamma \to \Gamma$ is defined by $G(v) = F(\mathscr{J} v)$, where $\mathscr{J} : \Gamma \to \overline{\mathscr{F}^w}$ is the nearest point map from Γ to the closed and convex subset $\overline{\mathscr{F}^w}$ to the uniformly convex space $L^p(\Omega)$, as in Proposition 5.14. It follows that F and G have the same fixed points: F and G agree on the set $\overline{\mathscr{F}^w}$ and elements of $\Gamma \setminus \overline{\mathscr{F}^w}$ lie outside of the ranges of both F and G, and so cannot be fixed points of either function.

For each $\beta > 0$ and $\emptyset \neq Z \subseteq \{1, \ldots, n\}$, set

$$U_Z^\beta = \{\zeta \in L^p(\Omega) : \mathrm{dist}(\zeta, \mathscr{R}_Z) < \beta\}.$$

By Lemma 5.30, U_Z^β are open sets where $V_Z^\beta = \partial U_Z^\beta \cap \overline{\mathscr{F}^w}$.

Fix $Z \in \mathscr{P}$. I seek some $\vartheta > 0$ such that

$$G(U_Z^\beta) \subset \mathscr{R}_Z \text{ and } d(\mathrm{id} - G, U_Z^\vartheta, 0) = 1.$$

For reference later in the proof, this ϑ will eventually be dropped.

Let B_i' denote the ball of radius r concentric with B_i, and $B_Z' = \bigcup_{i \in Z} B_i'$. For $f \in U_Z^\vartheta$, let $\mathscr{J}_Z f$ be the nearest point of $\mathscr{R}_Z$ to f; then

$$\|\mathscr{J}_Z f - f\|_p < \vartheta \text{ and } \|\mathscr{J} f - f\|_p < \vartheta \text{ which implies } \|\mathscr{J}_Z f - \mathscr{J} f\|_p < 2\vartheta.$$

Therefore,

$$\|K_\Omega \mathscr{J} f - K_\Omega \mathscr{J}_Z f\|_{\sup} \leq 2c_2\vartheta,$$

where c_2 is the norm of K_Ω as a linear operator from $L^p(\Omega)$ to $C(\overline{\Omega})$. Hence,

$$\int_\Omega (\mathscr{J}_Z f - k) \ \mathrm{d}\mu > n\lambda \text{ for } f \in U_Z^\vartheta,$$

because

$$\int_\Omega (g - k) \ \mathrm{d}\mu = \int_\Omega f_{00} \ \mathrm{d}\mu > n\lambda \text{ for all } g \in \overline{\mathscr{F}^w}.$$

It follows that

$$\text{for every } f \in U_Z^\vartheta \text{ there exists } i^* \in Z \text{ such that } \int_{B_{i^*}} (\mathscr{J}_Z f - k) \ \mathrm{d}\mu > \lambda. \tag{5.2}$$

For each $f \in U_Z^\vartheta$, it holds

$$\inf_{x \in B_{i^*}'} K_\Omega (\mathscr{J}_Z f - k) \ \mathrm{d}\mu \geq \alpha$$

with i^* as in (5.2), since $\|g - k\|_p \leq \|f_{00}\|_p = 1$ for all $g \in \overline{\mathscr{F}^w}$.

It may be further supposed that $2c_2\vartheta < 4^{-1}\alpha$, which ensures

$$\inf_{x \in B_{i^*}'} K_\Omega (\mathscr{J} f - k)(x) > \frac{3\alpha}{4} \text{ for all } f \in U_Z^\vartheta \text{ and } i^* \text{ as in (5.2)},$$

so it must be that

$$\inf_{x \in B'_{i^*}} K_\Omega \left(\mathscr{J} f\right)(x) > \frac{3\alpha}{4} \text{ for all } f \in U_Z^\vartheta \text{ and } i^* \text{ as in (5.2).}$$

Moreover, the choice of k ensures that, for each $x \in \Omega$

$$\begin{aligned} K_\Omega \left(\mathscr{J} f\right)(x) &= K_\Omega \left(\mathscr{J} f - k\right)(x) + k K_\Omega \left(\mathbb{1}_\Omega\right)(x) && \text{by linearity of } K \\ &< K_\Omega \left(\mathscr{J} f - k\right)(x) + \frac{\alpha}{4} \end{aligned}$$

and hence if $f \in U_Z^\vartheta$, then it follows that

$$K_\Omega \left(\mathscr{J} f\right)(x) < \frac{\alpha}{2} \text{ provided } \operatorname{dist}(x, \partial\Omega) < \varepsilon$$

by the application of Lemma 5.32.

Write $\psi = K_\Omega(\mathscr{J} f)$, so $\psi = \psi_1 + \psi_2 + k K_\Omega \left(\mathbb{1}_\Omega\right)$, where

$$\begin{aligned} \psi_1 &= K_\Omega \left((\mathscr{J} f - k)\mathbb{1}_{B_Z}\right) \\ \psi_2 &= K_\Omega \left((\mathscr{J} f - k)\mathbb{1}_{\Omega \setminus \overline{B_Z}}\right). \end{aligned}$$

Now, all points of ∂B_Z are within the distance ε of $\partial\Omega$, so

$$\psi_1 \leq K_\Omega \left(\mathscr{J} f - k\right) \leq \frac{\alpha}{4} \text{ on } \partial\Omega,$$

which implies

$$0 \leq \psi_1 \leq \frac{\alpha}{4} \text{ on } \partial\left(\Omega \setminus \overline{B_Z}\right).$$

Since ψ_1 is harmonic on $\Omega \setminus \overline{B_Z}$, the Maximum Principle implies that $0 \leq \psi_1 \leq \frac{\alpha}{4}$ on $\Omega \setminus \overline{B_Z}$. Also, by the Triangle Inequality and prior calculations,

$$\begin{aligned} \left\|\left(\mathscr{J} f - k\right)\mathbb{1}_{\Omega \setminus \overline{B_Z}}\right\|_p &\leq \left\|\left(\mathscr{J} f - \mathscr{J}_Z f\right)\mathbb{1}_{\Omega \setminus \overline{B_Z}}\right|_p + \left\|\left(\mathscr{J}_Z f - k\right)\mathbb{1}_{\Omega \setminus \overline{B_Z}}\right\|_p \\ &\leq \left\|\mathscr{J} f - \mathscr{J}_Z f\right\|_p \\ &< 2\vartheta, \end{aligned}$$

hence

$$\|\psi_2\|_{\sup} \leq 2c_2\vartheta < \frac{\alpha}{4}.$$

For each $x \in \Omega \setminus \overline{B_Z}$, it holds that

$$\psi(x) = \psi_1(x) + \psi_2(x) + kK_\Omega(\mathbb{1}_\Omega) < \frac{3\alpha}{4}.$$

Let $h = G(f) = F(\mathscr{J} f)$, so $h \in \mathscr{F}$ and $h = \varphi \circ \psi$ for an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$. It has already been calculated that $\psi(x) > \frac{3\alpha}{4}$ for all $x \in B'_{i^*}$ for some $i^* \in Z$, and $\psi(x) < \frac{3\alpha}{4}$ for all $x \in \Omega \setminus \overline{B_Z}$. The measure of the set where $h > k$ is at most $\mu(B'_{i^*})$, so it is concluded that $h - k$ vanishes on $\Omega \setminus \overline{B_Z}$ if $f \in U_Z^\vartheta$. Thus, $G(U_Z^\vartheta) \subseteq \mathscr{R}_Z$.

Thus, there exists $\vartheta > 0$ that satisfies the following three conditions:

(C1) for every $Z \in \mathscr{P}$, $G(U_Z^\vartheta) \subseteq \mathscr{R}_Z$;

(C2) $0 < \vartheta < \frac{1}{2} \min \left\{ \operatorname{dist}\left(\mathscr{R}_{\{i\}}, \mathscr{R}_{\{j\}}\right) : 1 \leq i < j \leq n \right\}$;

(C3) for every $Z \in \mathscr{P}$, ∂U_Z^ϑ contains no fixed points of G.

Suppose $\vartheta > 0$ satisfies $G(U_Z^\vartheta) \subseteq \mathscr{R}_Z$, then for every $0 < \vartheta' < \vartheta$, it holds that

$$G\left(U_Z^{\vartheta'}\right) \subset G\left(U_Z^{\vartheta}\right) \subseteq \mathscr{R}_Z.$$

The set of values that satisfies (C1) is an interval. By intersection, the set of values of ϑ that satisfies both (C1) and (C2) must form an interval in $\mathbb{R}$ with infimum 0. If no $\vartheta > 0$ meets the criterion (C3), then there are infinitely many fixed points of G, contravening one of the initial assumptions. Therefore, choose $\vartheta > 0$ that satisfies (C1) – (C3).

Schauder's Fixed Point Theorem utilises the convexity of $\mathscr{R}_Z$ and deduces the existence of a fixed point of G in $\mathscr{R}_Z$ and that G has no fixed points in the set $U_Z^\vartheta \setminus \mathscr{R}_Z$. Since

$$\bigcup_{W \in \mathscr{P}} U_W^\vartheta = U_{\{1,\ldots,n\}}^\vartheta = \left\{ \zeta \in L^p(\Omega) : \operatorname{dist}(\zeta, \overline{\mathscr{F}^w}) < \vartheta \right\},$$

there must be no fixed points of G outside of $\bigcup_{W \in \mathscr{P}} U_W^\vartheta$. For simplicity, suppress the upper index and call the open set $U_Z \subset \Gamma$. Now, choose a fixed point $u^* \in U_Z$.

Topological degree theory can be applied to this boundary value problem, and I seek to show that

$$0 \notin \left[\mathrm{id} - u^* - t(G - u^*)\right](\partial U_Z) \text{ for all } t \in [0, 1].$$

Assume, seeking contradiction, there exist $f \in \partial U_Z$ and some $t \in [0, 1]$ such that

$$f - u^* - t(G(f) - u^*) = 0.$$

Expressing this equation in terms of f yields:

$$f = tG(f) + (1-t)u^*.$$

The convexity of U_Z implies that $f \in U_Z$, but $f \in \partial U_Z$. However, $U_Z \cap \partial U_Z = \emptyset$, so this is a contradiction. The homotopy

$$H : \overline{U_Z} \times [0,1] \to \overline{\mathscr{F}^w},\ H(f,t) = f - tG(f) - (1-t)u^*$$

between $\mathrm{id} - u^*$ and $\mathrm{id} - G$ is valid over $\overline{U_Z}$, so (d3) may be applied. Hence,

$$\begin{aligned} d(\mathrm{id} - G, U_Z, 0) &= d(\mathrm{id} - u^*, U_Z, 0) && \text{by (d3)} \\ &= d(\mathrm{id}, U_Z, u^*) && \text{by (d4)} \\ &= 1 && \text{by (d1).} \end{aligned}$$

Colloquially, this means that the fixed points identified so far have "total degree 1".

Now, it is required that further fixed points are found, so I use degree theory to demonstrate that there are fixed points of the maximiser function F concentrated in any non-empty sub-collection of balls. I seek to show that for every $Z \in \mathscr{P}$ with $|Z| = m$, there is at least one fixed point of G concentrated in B_Z – that is, the pre-image of the set (k, ∞) is a subset of D_Z up to sets of zero Lebesgue measure – and that

$$d\left(\mathrm{id} - G, U_Z \setminus \bigcup_{\substack{\emptyset \neq W \subset Z \\ W \neq Z}} \overline{U_Y}, 0\right) = (-1)^{m+1}.$$

Induction will be used to prove this claim. The base case follows from the above calculations with $Z = \{i\}$ for $i \in \{1, \ldots, n\}$. Note that there are also no smaller sub-collections than a collection comprising a single ball.

Assume, for the principle of finite induction, there exists $M \in \mathbb{N}$ and $M < n$ such that the claim is true for all $1 \leq m \leq M$. I seek to show that the claim is still valid for $m = M+1$. In a ball collection of size $M+1$, the number of sub-collections of j balls is the binomial coefficient $\binom{M+1}{j}$. By the inductive hypothesis, each sub-collection of size m has at least one fixed point with "total degree" $(-1)^{m+1}$ where the pre-image of the set (k, ∞) is a subset of the union of the specified B_i, up to sets of zero Lebesgue measure zero. Let $Z \in \mathscr{P}$, and define

$$U_Z^\dagger = U_Z \setminus \bigcup_{\substack{\emptyset \neq W \subset Z \\ W \neq Z}} \overline{U_W} \subseteq \Gamma,$$

where an empty union is equal to $\emptyset$. For $i \in \{1, \dots, n\}$, $U_{\{i\}}^\dagger = U_{\{i\}}$. By an inductive argument on $|Z|$, it can be shown that

$$U_Z^\dagger = U_Z \setminus \bigcup_{\substack{\emptyset \neq W \subset Z \\ W \neq Z}} \overline{U_W^\dagger}.$$

This open set $U_Z^\dagger$ contains at least one fixed point of G, and the total degree over the whole set is

$$d(\mathrm{id} - G, U_Z^\dagger, 0) = (-1)^{M+1}.$$

The condition (C2) on ϑ implies that $U_{\{i\}} \cap U_{\{j\}} = \emptyset$ for $1 \leq i < j \leq n$. If $W, Z \in \mathscr{P}$ are disjoint, then U_W and U_Z are disjoint. Consequently, if $W, Z \in \mathscr{P}$ and $W \neq Z$ then $U_W^\dagger \cap U_Z^\dagger = \emptyset$. For measurable subsets A, B of a topological space, it is well-known that

$$\partial(A \setminus B) \subseteq \partial A \cup \partial B.$$

Hence,

$$\partial U_Z^\dagger = \bigcup_{\emptyset \neq W \subseteq Z} \partial U_W.$$

By the choice of ϑ in (C3), there are no solutions of $\mathrm{id} - G = 0$ in ∂U_W, for every $\emptyset \neq W \subseteq Z$. Hence, there are no solutions of the equation $\mathrm{id} - G = 0$ in $\partial U_Z^\dagger$. These open sets are always disjoint for distinct, non-empty subsets of $\{1, \dots, n\}$, and contain no solutions of the equation $\mathrm{id} - G = 0$ on their boundaries. This means that (d5) may be applied over the following collection of open sets: $\left\{ U_W^\dagger : W \in \mathscr{P},\ W \subseteq Z \right\}$. It follows that

$$\begin{aligned}
1 &= d(\mathrm{id} - G, U_Z, 0) && \text{by prior calculation} \\
&= d(\mathrm{id} - G, U_Z^\dagger, 0) + \sum_{\substack{\emptyset \neq W \subset Z \\ W \neq Z}} d(\mathrm{id} - G, U_W^\dagger, 0) && \text{by (d5)} \\
&= d(\mathrm{id} - G, U_Z^\dagger, 0) + \sum_{j=1}^{M} (-1)^{j+1} \binom{M+1}{j} && \text{by the inductive hypothesis} \\
&= d(\mathrm{id} - G, U_Z^\dagger, 0) + 1 + (-1)^{M+1} && \text{by the Binomial Theorem.}
\end{aligned}$$

Consequently,

$$d(\mathrm{id} - G, U_Z^\dagger, 0) = (-1)^{M+2} \neq 0.$$

By an application of (d2), there must exist $u_Z \in U_Z^\dagger$ such that $G(u_Z) = F(u_Z) = u_Z$. Note that each fixed point u_Z must be distinct from the other identified fixed points, which are all concentrated in ball sub-collections of size M or less. Thus, the claim is true for $m = M + 1$, and so, by the principle of mathematical induction, must hold for all $m \leq n,\ m \in \mathbb{N}$.

There are now two methods for realising the final result. Either, consider the number of fixed points identified by the above process:

$$\sum_{j=1}^{n-1} \binom{n}{j} + 1 = (2^n - 2) + 1 = 2^n - 1.$$

Alternatively, note that there exists one fixed point for each element of $\mathscr{P}$, which is a set of size $2^n - 1$. □

A simple corollary is where, rather than approximating the union of n disjoint open balls, the domain is actually equal to the union of n disjoint open balls.

Corollary 5.35. *Let $0 < r < r_1 \leq \ldots \leq r_n$ for some $n \in \mathbb{N}$; and suppose $D_1, \ldots, D_n \subset \mathbb{R}^N$ are balls with disjoint closures with respective radii $r_1, \ldots, r_n$. Let $0 < \lambda < \frac{1}{n}\mu(\mathbb{B}(0, r_1))^{\frac{1}{q}}$ be given, $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$, $p > \frac{N}{2}$ and $N \geq 2,\ N \in \mathbb{N}$. Let $\alpha = \alpha_0$, which is provided by Lemma 5.31. Let $D_1, \ldots, D_n \subset \mathbb{R}^N$ be balls with disjoint closures having respective radii $r_1, \ldots, r_n$, let V_0 be a right circular cone with vertex 0, let $\varepsilon > 0$ be the number provided by Lemma 5.32, and let $k_0 > 0$ be the positive number provided by Lemma 5.33.*

Suppose Ω is equal to the union of $D_1, \ldots, D_n$, let $0 \leq f_{00} \in L^p(\Omega)$ be a function vanishing outside of a set of measure $\mu(\mathbb{B}(0, r))$ and satisfying the integral condition

$$0 < \lambda n \left\| f_{00} \right\|_p < \int_\Omega f_{00}\ d\mu.$$

Let $0 < k \leq k_0$ and set

$$f_0 : \Omega \to \mathbb{R},\ f_0(x) = k + f_{00}(x).$$

Let $\mathscr{F} \subset L^p(\Omega)$ be the set of rearrangements of f_0 on Ω, and let $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ be the maximiser function defined in Definition 5.10. Then F has at least $2^n - 1$ fixed points and they all belong to $\mathscr{F}$.

Proof. Note that Ω satisfies $(B1)-(B4)$ relative to concentric balls $D_1^\dagger, \ldots, D_n^\dagger$. Theorem 5.34 can be applied, yielding the existence at least $2^n - 1$ fixed points for F. $\square$

Remark 5.36. In the case where $n = 1$, there is also one fixed point – as shown by Lemma 5.27, which is concordant with Corollary 5.35.

These results should be interpreted physically. Suppose u is a function that satisfies $-\Delta u = \varphi \circ u$ in a bounded domain Ω for a sufficiently smooth function φ, and $u = 0$ on $\partial\Omega$. This function represents the stream function for the steady flow of an ideal fluid in N dimensions, where $\partial\Omega$ represents the shape of the wall bounding the fluid. This wall may be considered to be piping in two or three dimensions.

The vorticity of this fluid is the curl of the velocity, and it has magnitude $-\Delta u$. Theorem 5.34 and Corollary 5.35 seeks flows where the vorticity is a rearrangement of a given function. In these results, the domain Ω is approximately equal to n disjoint balls in $\mathbb{R}^N$, and at least $2^n - 1$ solutions of the boundary value problem involving rearrangements are shown to exist. These solutions correspond to distinct arrangements of a region of non-zero vorticity in a flow where the vorticity is otherwise zero, that is, irrotational.

These results say that a linear increase in the number of disjoint balls leads to an exponential increase in the multiplicity of solutions to the given boundary value problem. This suggests that piping will deteriorate exponentially through usage and dents.

Example 5.37. Given the domain Ω satisfies $(B1) - (B4)$, it can be said that Ω approximates the union of n balls. The topological degree theory method does not distinguish between a domain that has no joining channels between the balls and one that has several. Suppose Ω satisifes $(B1) - (B4)$ with respect to two balls $D_1, D_2 \subset \mathbb{R}^N,\ N \geq 2$, and has two channels joining those balls, which are denoted C_1 and C_2. According to Theorem 5.34, the maximiser function has at least three fixed points.

By construction, the channels have small measures so that $\mu(C_1) + \mu(C_2) < \mu\left(\{x \in \Omega : f_0(x) > k\}\right)$. This means that there are no rearrangements that can be wholly concentrated in the two channels. Set

$$
\begin{aligned}
\mathscr{R}_{(1)} &= \left\{g \in \mathscr{F} : g^{-1}(k,\infty) \subseteq D_1\right\}, \\
\mathscr{R}_{(2)} &= \left\{g \in \mathscr{F} : g^{-1}(k,\infty) \subseteq D_2\right\}, \\
\mathscr{R}_{(3)} &= \left\{g \in \mathscr{F} : g^{-1}(k,\infty) \subseteq D_1 \cup D_2\right\}, \\
\mathscr{R}_{(4)} &= \left\{g \in \mathscr{F} : g^{1}(k,\infty) \subseteq D_1 \cup D_2 \cup C_1\right\}, \\
\mathscr{R}_{(5)} &= \left\{g \in \mathscr{F} : g^{-1}(k,\infty) \subseteq D_1 \cup D_2 \cup C_2\right\}, \\
\mathscr{R}_{(6)} &= \left\{g \in \mathscr{F} : g^{-1}(k,\infty) \subseteq D_1 \cup D_2 \cup C_1 \cup C_2\right\},
\end{aligned}
$$

where these inclusions hold up to sets of Lebesgue measure zero.

Similarly to Theorem 5.34, set $U_{(i)} = \left\{f \in \mathscr{F} : \operatorname{dist}(f, \mathscr{R}_{(i)}) < \beta\right\}$ for sufficiently small $\beta > 0$ and $i \in \{1, \ldots, 6\}$. Note

$$U_{(1)}, U_{(2)} \subseteq U_{(3)} \subseteq U_{(4)}, U_{(5)} \subseteq U_{(6)}.$$

As in Theorem 5.34, it can be calculated that

$$d(\mathrm{id} - G, U_{(i)}, 0) = 1 \text{ for } i \in \{1, \ldots, 6\}.$$

Denote the disjoint open sets, similarly to Theorem 5.34, by

$$
\begin{aligned}
U^{\dagger}_{(i)} &= U_{(i)} && \text{for } i \in \{1,2\} \\
U^{\dagger}_{(3)} &= U_{(3)} \setminus \left(\overline{U_{(1)}} \cup \overline{U_{(2)}}\right) \\
U^{\dagger}_{(j)} &= U_{(j)} \setminus \overline{U_{(3)}} && \text{for } j \in \{4,5\} \\
U^{\dagger}_{(6)} &= U_{(6)} \setminus \left(\overline{U_{(5)}} \cup \overline{U_{(6)}}\right).
\end{aligned}
$$

It is clear that

$$d(\mathrm{id} - G, U^{\dagger}_{(i)}, 0) = d(\mathrm{id} - G, U_{(i)}, 0) = 1 \text{ for } i \in \{1,2\}.$$

It is immediate that

$$d(\mathrm{id} - G, U^{\dagger}_{(3)}, 0) = 1 - d(\mathrm{id} - G, U_{(1)}, 0) - d(\mathrm{id} - G, U_{(2)}, 0) = -1.$$

A short calculation yields, for $j \in \{4,5\}$,

$$\begin{aligned} d(\mathrm{id}-G, U^\dagger_{(j)}, 0) &= d(\mathrm{id}-G, U_{(5)}, 0) - \sum_{i=1}^{3} d(\mathrm{id}-G, U^\dagger_{(i)}, 0) && \text{by } (d5) \\ &= 1 - (1+1-1) && \text{by prior calculation} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} d(\mathrm{id}-G, U^\dagger_{(6)}, 0) &= d(\mathrm{id}-G, U_{(6)}, 0) - \sum_{i=1}^{5} d(\mathrm{id}-G, U^\dagger_{(i)}, 0) && \text{by } (d5) \\ &= 1 - (1+1-1+0+0) && \text{by prior calculation} \\ &= 0. \end{aligned}$$

Hence, degree theory only identifies three fixed points of G, and hence F. This shows that the lower bound of the multiplicity of fixed points in Theorem 5.34 is *not* increased by the presence of more channels.

It should be demonstrated why the degree theory method is more powerful than variational methods.

Corollary 5.38 (Domain is Rotationally Symmetric)**.** *Suppose $\Omega \subset \mathbb{R}^2$ and $f_0 : \Omega \to \mathbb{R}$ both satisfy the conditions of Theorem 5.34 with respect to a collection of three discs in $\mathbb{R}^2$. Suppose further that Ω is rotationally symmetric with respect to a rotation of $\frac{2\pi}{3}$ radians around a point in $\mathbb{R}^2$. Denote the set of rearrangements of f_0 on Ω by $\mathscr{F}$, where $f_0 \in L^p(\Omega)$ for some $1 < p < \infty$, then the maximiser function $F : \overline{\mathscr{F}^w} \to \overline{\mathscr{F}^w}$ has at least seven fixed points.*

Proof. Whilst it is immediate as a corollary that F has at least seven fixed points, the differences that can be made in calculating that answer should be elaborated upon.

Let the discs be labelled as D_1, D_2 and D_3, and call the rotation of $\frac{2\pi}{3}$ radians around a point in $\mathbb{R}^2$ R, so $R(D_1) = D_2$, $R(D_2) = D_3$ and $R(D_3) = D_1$. Using the notation in Theorem 5.34, only the degree of $\mathrm{id} - G$ of the sets $U^\dagger_{\{1\}}$ and $U^\dagger_{\{1,2\}}$ at 0 need to be directly calculated. Note that $U^\dagger_{\{2\}} = U^\dagger_{\{1\}} \circ R$, $U^\dagger_{\{3\}} = U^\dagger_{\{2\}} \circ R$ and $U^\dagger_{\{2,3\}} = U^\dagger_{\{1,2\}} \circ R$, $U^\dagger_{\{1,3\}} = U^\dagger_{\{2,3\}} \circ R$.

By Theorem 5.21 (iv):

$$\begin{aligned}d(\mathrm{id}-G,U^{\dagger}_{\{1\}},0)&=d(\mathrm{id}-G,U^{\dagger}_{\{2\}},0)=d(\mathrm{id}-G,U^{\dagger}_{\{3\}},0)=1\\ d(\mathrm{id}-G,U^{\dagger}_{\{1,2\}},0)&=d(\mathrm{id}-G,U^{\dagger}_{\{2,3\}},0)=d(\mathrm{id}-G,U^{\dagger}_{\{1,3\}},0)=-1\\ d(\mathrm{id}-G,U^{\dagger}_{\{1,2,3\}},0)&=d(\mathrm{id}-G,\Gamma,0)=1.\end{aligned}$$

Since each U_Z contains at least one fixed point of F, these sets are disjoint and there are seven such sets, F must have at least seven fixed points. □

Remark 5.39. A proof of Corollary using variational methods would first identify the fixed points of F concentrated in the three discs. Then, it would examine paths of rearrangements that lie between those three fixed points, labelled u_1, u_2 and u_3, where $u_2 = u_1 \circ R$ and $u_3 = u_2 \circ R$. Set

$$\Psi : L^p(\Omega) \to \mathbb{R},\ \ \Psi(\zeta) = \int_{\Omega} \zeta K \zeta \ \mathrm{d}\mu.$$

For $i, j \in \{1, 2, 3\}$ and $i < j$:

$$\begin{aligned}\mathscr{C}_{i,j} &= \{h \in C([0,1], \mathscr{F}) : h(0) = u_i,\ \ h(1) = u_j\}\\ c_{i,j} &= \sup_{h\in\mathscr{C}_{i,j}} \inf_{t\in[0,1]} \Psi(h(t)).\end{aligned}$$

It can be shown that

$$c_{1,2} = c_{2,3} = c_{1,3}.$$

There are two ways to demonstrate this result. Without loss of generality, I focus on $c_{1,2}$ and assume $c_{1,2} \leq \min\{c_{1,3}, c_{2,3}\}$. Notice that paths running joining u_1 and u_2 in $\mathscr{F}$ may also pass through u_3, that is,

$$\left\{h \in \mathscr{C}_{1,2} : h\left(\frac{1}{2}\right) = u_3\right\} \subseteq \mathscr{C}_{1,2}.$$

It follows that

$$\sup_{h\in\mathscr{C}_{1,3}\cup\mathscr{C}_{2,3}} \inf_{t\in[0,1]} \Psi(h(t)) \leq \sup_{h\in\mathscr{C}_{1,2}} \inf_{t\in[0,1]} \Psi(h(t)),$$

and applying the notation yields

$$\max\{c_{1,3}, c_{2,3}\} \leq c_{1,2}.$$

Combining this inequality with the assumption that $c_{1,2} \leq \min\{c_{1,3}, c_{2,3}\}$ means

$$c_{1,2} = c_{2,3} = c_{1,3}.$$

Alternatively, the rotation R around a point by $\frac{2\pi}{3}$ radians can be applied directly to the sets of paths:

$$\mathscr{C}_{1,2} \circ R = \mathscr{C}_{2,3} \text{ and } \mathscr{C}_{2,3} \circ R = \mathscr{C}_{1,3}.$$

Also, for $h \in \mathscr{C}_{1,2}$ and $t \in [0,1]$:

$$\begin{aligned}
\Psi((h \circ R)(t)) &= \int_\Omega (h \circ R)(t) K (h \circ R)(t) \, \mathrm{d}\mu \\
&= \int_\Omega (h(t) \circ R) K (h(t) \circ R) \, \mathrm{d}\mu && \text{by definition} \\
&= \int_\Omega (h(t) \circ R)(K h(t)) \circ R \, \mathrm{d}\mu && \text{by Lemma 5.20} \\
&= \int_\Omega h(t) K h(t) \, \mathrm{d}\mu && \text{as } R \text{ is measure-preserving} \\
&= \Psi(h(t)).
\end{aligned}$$

It is an immediate consequence that

$$c_{1,2} = c_{2,3} = c_{1,3}.$$

For this reason, a naive variational methods would be unable to distinguish between the fixed points of F generated by examining the paths in between a pair of the first three fixed points. It may be possible to discover the fourth, fifth and sixth fixed points by restricting the set of paths to only those rearrangements nearly concentrated in each pair of discs. However, I am unaware of any variational methods that can produce the seventh fixed point found in Corollary 5.39.

Finally, I note that the norm used in Theorem 5.34 was not specified.

Corollary 5.40. *The results of Theorem 5.34 hold for any norm in* $\mathbb{R}^N$.

Proof. On a finite-dimesional vector space such as $\mathbb{R}^N$, it can be shown that all the norms are equivalent. Hence, Theorem 5.34 and the preceding lemmas holds for any norm on $\mathbb{R}^N$. □

This means that in the previous results, the domain may be approximate to a union of squares, cubes and other shapes defined by other finite-dimensional norms, rather than just balls defined by the Euclidean norm.

Chapter 6

Conclusions and Further Work

With regards to the results concerning the use of topological degree theory in fluid flow problems, it would be desirable to extend the work provided by Theorem 5.34. Currently, the result only applies to domains that approximate the union of a given number of balls, with restrictions on what the base function for the set of rearrangements is. Since all norms on finite-dimensional real space are equivalent, these "balls" are not necessarily balls with respect to the 2-norm on $\mathbb{R}^N$. Certainly, it is possible that this topological degree method may be applied to other boundary value problems, rather than those arising specifically from fluid flow problems. Theorem 5.34 has demonstrated the slick power of the topological degree method, providing an increase in the multiplicity of solutions to the given boundary value problem. Whilst variational methods may have to be tailored to each problem, the only requirement on the topological degree method is that the degree over numerous open subsets of the function space Γ have to be calculated. This is often repetitive, but once these are calculated, the topological degree method swiftly provides the existence of distinct fixed points. In the case that the domain is the union of n balls, this means there are at least $2^n - 1$ fixed points, which are equivalent to $2^n - 1$ solutions of the given boundary value problem. It is unlikely that the restriction on the exponent p will be able to be weakened, since this exponent provides key elliptic regularity results from [36]. The general relationship between the domain's shape and the number of solutions to the related boundary value problem, in the vein of [19, 20], should be studied.

A major restriction on the results given in this thesis is that the domain is bounded. Flow in unbounded domains should be of vital consideration, but will require the generalisation of rearrangements to measure spaces with infinite measure. R. J. Douglas provided this generalisation in the paper [24]. It would be satisfying to see the properties of the set of rearrangements, and other properties like rearrangement inequalities, proven for infinite measure spaces. A subsequent investigation would be what are the minimal assumptions placed on the measure space to ensure that rearrangements and the set of rearrangements have particular properties, such as contractibility. The relationship between structural assumptions, such as the domain being a non-compact connected semisimple Lie groups in [43], and desirable properties of rearrangements and the set of rearrangements should also be investigated. In the case of [43], this property is a generalisation of the Riesz rearrangement inequality.

Currently, the topological degree theory method is lacking a classification theorem for the fixed points of the maximiser function, as seen in work by Hofer [42]. The main claim of that paper is the topological degree of isolated critical points given by the Ambrosetti-Rabinowitz Mountain-pass Theorem in [5] are -1. This has an analogy with the work provided in the thesis, where the fixed point, if isolated, that lies on the mountain pass between two fixed points concentrated in two balls, must have Leray-Schauder degree -1. The generalised Mountain Pass Lemma may be further extended to higher-dimensional spanning surfaces.

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